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ABSTRACT

The main purpose of this special issue is to suggest strategies for meeting the mathematical needs of two-year college students in the decade of the 1990s. Since the 20 contributors were free to write about topics that they considered to be appropriate to the intellectual and personal development of all students their papers represent a broad overview of what leading mathematics educators think should be occurring in mathematics education in the 1990s. The articles are separated into the following five broad categories: (1) guiding thoughts about technological innovation, curriculum standards, and concomitant implementation; (2) outlines of curriculum content specific to calculus, discrete mathematics, statistics, and fractal geometry; (3) special considerations regarding remedial coursework and the needs of the learning disabled student; (4) utilization of computer algebra systems, videotapes, and multimedia presentations in the college mathematics classroom; and (5) the content and reasonable uses of placement tests. Each author supplies diagrams and references as appropriate. (JJK)

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THE AMATYC REVIEW

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first there were

PYTHAGORAS ARISTOTLE EUCLID PTOLEMY

then came

DESCARTES NEWTON COPERNICUS
EULER LIEBNITZ LAGRANGE
PASCAL ABEL NAPIER TAYLOR GIBBS

and now

TWO-YEAR COLLEGE MATHEMATICS EDUCATION FOR THE 1990s

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
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About the Cover

The painting on the front cover is a portion of the *School of Athens* painted by Raphael in 1508-11. It is one of a series of murals in the papal apartments of the Vatican.

The painting is a study of the great philosophers and mathematicians of Athens and it symbolizes the spirit of investigation and thinking. On the top step Aristotle talks to Plato. In the foreground (from the left), Pythagoras consults on a book. At his feet is held a diagram of the Pythagorean harmonic scale. On the far right Euclid expounds his theories, while Ptolemy, looking on, holds a globe.

Cover design by Robert Burghardt

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The AMATYC Review

Special Issue

Two-Year College Mathematics Education for the 1990s: A Commitment to Quality

Edited by

Don Cohen
State University of New York
College of Agriculture and Technology
Cobleskill, NY 12043

October, 1989

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Karl J. Smith, President
Santa Rosa Junior College
Santa Rosa, CA 95401

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FORWARD

Don Cohen
Editor, The AMATYC Review
SUNY College of Agriculture and Technology
Cobleskill, NY 12043

The title of an article in a recent issue of *The Chronicle of Higher Education* really caught my attention. John Kelley (1989) proposes that "Colleges Shouldn't Waste Their Resources on Students Who are not Qualified to Be There in the First Place." He believes that colleges should not "undertake the thankless task of trying to force learning on students who are not prepared to make the personal commitment or whose need for remediation is so extreme that they have no hope of succeeding at the college level" (p. B2). He would rather see a reduction in the number of institutions of higher learning than to have colleges enroll students who are not eager for knowledge.

On the other hand, John Gardner (1984), in his wonderful book, *Excellence*, takes a more positive view toward educating students at all ability levels. He sees the practice of open admissions to all high school graduates that is followed by many colleges as providing equality of opportunity in our democratic society. Gardner argues that "It is better to let students try and fail – and in failing discover their own inadequacy – then to tell them they are not good enough to try" (p. 83).

As a two-year college educator, I certainly agree with the principle of equality of opportunity. However, when colleges enroll students without making extensive efforts to provide the necessary educational support so that all students have a "realistic" opportunity to be successful, we might as well adopt Kelley's position and not enroll high risk students in the first place.

I am not suggesting that we "water-down" our programs. Colleges must establish challenging programs that are appropriate for all students and then demand that they strive to learn. Gardner (1984) states: "It is no sin to let average as well as brilliant youngsters into college. It is a sin to let any substantial portion of them – average or brilliant – drift through college without effort, without growth, and without a goal" (p. 101). Those students who do not have the necessary academic ability to profit from a quality educational program or those who choose not to learn should not be allowed to remain in college.

Joseph Burke (1988), Provost of the State University of New York states: "... the true test of institutional quality of a school should be its contribution, or value-added, to the intellectual or personal development of its students from admission to graduation," (p. 4). Furthermore, Burke explained:

> No public university should accept the aristocratic assumption that the only entrée to excellence is through exclusivity – that excellence is only achievable by limiting access. Neither should a public university accept the contrary thesis that educational inferiority is the inevitable consequence of access.... A value-added approach to assessment combines access with excellence by insisting that each student and each institution become the best they can be. (p. 5)

In summary, to paraphrase John Kelley, colleges must *spend* their resources to provide the opportunity for a quality education for all citizens who are ready and willing. Two-year colleges are the institutions of higher learning that have the opportunity to enroll students with a wide variety of abilities and life-styles and these institutions must be committed to meeting the needs of their students.

This Special Issue

The main purpose of this special issue of *The AMATYC Review*, "Two-Year College Mathematics Education for the 1990s: A Commitment to Quality," is to suggest strategies for meeting the mathematical needs of our students in the coming decade. The articles are separated into five sections: Guiding Thoughts, Curriculum, Special Programs, Technology, and Placement. Since the authors were free to write about topics they thought

to be appropriate, the readers will get a broad overview of what leading mathematics educations think *should* be occurring in mathematics education in the 90s.

Since there are no articles that explicitly address the transfer of mathematics majors, I asked Nelson Rich of Nazareth College (a four-year liberal arts college in Rochester, NY) for this thoughts on this topic. (Nelson is a former two-year college mathematics teacher and is presently a member of our editorial panel.) At Nazareth, while course titles are not likely to change, Nelson sees a shift in emphasis within required courses. There is an expanding emphasis on the use of Computer Algebra Systems (CAS) and hand-held calculators with sophisticated graphing capabilities. He believes that it is inevitable that as software becomes available on less expensive hardware its integration into the calculus sequence and elsewhere will be pervasive. If two-year college students are to be prepared for transfer, then technology must be fully integrated into their two-year college mathematics program.

Arnold Pizer, the Associate Chairman of the Mathematics Department at the University of Rochester (NY) also believes that there will not be major changes in the necessary subject matter background for students transferring to his institution. He urges two-year college mathematics departments to concentrate on the quality of their courses rather on the quantity.

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The department of mathematics at Oregon State University is currently involved in a calculus curriculum project with support provided by the National Science Foundation and Hewlett-Packard. The major goal of the project is to develop calculus curricular materials which make effective pedagogical use of technology, particularly "super calculators" such as the HP-28S.

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Section 1.
Guiding Thoughts

- Reconceptualizing Mathematics in the context of Technological Innovation – by Marvin L. Johnson

  "Our goal must be a reconceptualization of mathematics instruction. This reconceptualizing must include the integration of technology into instruction, but in a way which encourages conceptual understanding and not rote memorization of procedures in a mindless search for the answer."

- Challenge for the 90's – by Karl J. Smith

  "Educators often try to solve problems by forming study groups or task forces. They are fond of making policy recommendations. But if what I say here is to make any difference, it must go beyond policy recommendation. It must motivate you to be a mover and a shaker, and must force you to take action. The answer rests with you – not someone else."

- Whither Two-Year College Mathematics – By Anthony Ralston

  "But the basic message is clear; you **must** aim toward using computers in the classroom (and outside of it, too, of course) or you are going to shortchange your students and you will miss a wonderful opportunity to make them like mathematics more than they do now."

- The New Standards for School Mathematics and Their Implications for College Mathematics – by David E. Kullman

  "...it is expected that the *standards* will help mold a new breed of mathematics student. As more students take a third and fourth year of mathematics in high school, college enrollments in algebra and remedial mathematics should decline."

- Mathematics and the Naval Officer – by Richard F. Maraszewski

  "It is also quite obvious that the technological environment that we have alluded to is not static, but rapidly changing. Therefore, our midshipman must also 'learn how to learn.' There is no better major to accomplish this goal than a mathematics major."

Reconceptualizing Mathematics in the Context of Technological Innovation

Marvin L. Johnson

College of Lake County, Grayslake, IL 60030

Recently the mathematics curriculum has come under fire at all levels. The elementary school has been criticized for its emphasis on drill and the teaching of the arithmetic algorithms in unimaginative ways. Tools such as calculators and computers which could help ameliorate this situation are chronically underused in most classrooms. Innovative teaching, lab work and special projects remain disappointingly rare (*TIME*, June 20, 1988, p. 79).

The mathematics achievement of secondary school students has also been roundly criticized. Studies have indicated that the mathematical achievement level of U.S. students is substantially below that of their Japanese and European counterparts (McKnight, et al., 1987; Baldwin, 1988).

This lack of achievement has begun to affect collegiate mathematics. Many young people arriving at college lack even basic computational skills (Boyer, 1987, p. 3). This has caused a substantial change in college mathematics offerings. The principal objective of mathematics instruction at the community college, the primary entrance point to higher education for underprepared and disadvantaged students, has become remediation of the poor preparation of high school graduates (Cohen & Brawer, 1987, p. 40). In six urban community college districts 60% of the mathematics enrollment was in remedial courses (Cohen & Brawer, 1987, p. 32). Two-year colleges are not alone in this situation. The Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America has stated it quite simply, "Too many people know too little mathematics" (Wardrop & Wardrop, 1982). This fact obviously reduces students' chances of successfully completing mathematics courses and also seriously affects their ability to deal with courses with mathematical prerequisites.

College level mathematics has recently had its turn in the pillory of criticism because of poor student performance in the initial college level mathematics course, calculus. Nationwide, approximately one million students enroll in calculus each year, 700,000 of them at the collegiate level. Fewer than 25% survive to enter the pipeline of careers in mathematics, science and engineering (Steen, 1987, p. xi, p. 10). Additionally, surveys indicate that many calculus classes consist of lecture sections of 250 or more students (Kolata, 1987, p. 91). Much of the instruction is relegated to teaching assistants who are poorly trained, indifferent to the course and may even have a limited mastery of the English language (Kolata, 1987, p. 92). Innovations in instruction are few. A survey conducted by the Conference Board of Mathematical Sciences revealed that only 7% of calculus courses used computers (Albers, 1987).

A Climate for Change

With these deficiencies in mathematics education as a backdrop, several current factors are creating a climate for change in the mathematics curriculum. First, the presence of calculators and symbolic manipulation computer software packages offers a way to de-emphasize computation in favor of emphasizing concepts. Second, public awareness of the importance of mathematics in a technological society is increasing and can be harnessed to provide more resources for mathematics instruction. Third, there is a movement in the mathematics community to reform calculus instruction and attempt to improve the dismal nationwide completion statistics in this course. Finally, the examination of the calculus has led to the examination of courses prior to and subsequent to calculus.

Using Technology

Our current mathematical modus operandi has been excoriated by Lynn Steen (Chronicle of Higher Education, Oct. 14, 1987). He feels that we can no longer tolerate the situation which has allowed college mathematics to stay in the "paper and pencil era." We must allow the available technology to assist us in teaching mathematical concepts. Steen argues that freeing students from computational drudgery via the new technology will allow us to emphasize concepts instead of computations. This will be a tremendous democratizing in-

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Marvin received a bachelors degree in mathematics from Wheaton College (IL), a masters degree in mathematics from Oakland University (MI) and a PhD in evaluation research from the University of Illinois—Chicago. He has taught at the College of Lake County since 1973.

overstated. The use of these devices will not be a panacea. Mindless button pushing of calculators and computers is no better than mindless application of paper and pencil algorithms. As an example of what can occur, we need only examine the vast quantity of poor quality research resulting from inappropriate use of canned statistics computer packages. When statistical tests have been performed simply because the test was included in a computer package and the data was available, the results have usually been dubious. When such tests are done without regard for the suitability of the technique, it is clear that merely taking the drudgery out of computing does not guarantee that something of value will be produced.

A Reconceptualization of Mathematics

Our goal must be a **reconceptualization** of mathematics instruction. This reconceptualization must include the integration of technology into instruction, but in way which encourages conceptual understanding and not rote memorization of procedures in a mindless search for the "answer." In order to accomplish this reconceptualization, five factors must be considered: the curriculum, the standardized tests used for evaluation, the teacher, the textbook, and the student.

Stephen Willoughby, former President of the National Council of Teachers of Mathematics observed in 1984 that "the most important factor in determining what mathematics is taught is the textbook used" (Nicely, 1985). In light of this observation, the curriculum and the textbook are virtually equivalent. Indeed, it can safely be said that our curriculum at all levels is driven by the available textbooks. Therefore, a key element in the reconceptualization of mathematics is the production of a new generation of textbooks. These must be books that do more than pay lipservice to modern technology. That is, these books cannot simply have some problems at the end of each section labeled "calculator or computer required." The new technology must be integrated into the text rather than forced on it. Since such texts appear to be unavailable at present, this means that teachers need to insist upon such texts from publishers and write such texts themselves. The order of these two steps is important because in today's competitive publishing world, publishers will not risk capital in the development of new texts unless they see a market for them.

A second factor driving the current curricula is the number of standardized tests used at all levels in our educational system. Among these tests are: the California Achievement Test (CAT), the Iowa Test of Basic Skills (ITBS), the Advanced Placement Tests for Calculus (AP), the Scholastic Aptitude Test (SAT), the American College Test (ACT) and the Graduate Record Exam (GRE). I believe that we teach mathematics courses with these types of tests in mind to a greater degree than we care to admit.

None of these tests permits an examinee to use a calculator. One reason for this is fairness. It would not be fair for some students to have access to these aids while others are denied them because of economic limitations. This is certainly not the case today since modern technology has brought the price of calculators within the reach of all students, particularly those at the college level. For this reason, I believe that the fairness argument is no longer valid. Another probable, though less discussed reason for the exclusion of calculators, is the presence of strictly computational types of questions on standardized tests. Test manufacturers believe that to allow calculators would render such questions meaningless. In my view, only elementary level tests should contain strictly computational items. Tests given to high school and college students should be more conceptually oriented.

Therefore, I propose that the use of calculators be allowed on all standardized tests at the secondary and post-secondary level. It should be a priority of our national mathematics organizations to work with the testing agencies to accomplish this change. This proposal is not radical, it is simply a logical extension of the current policy of organizations such as the National Council of Teachers of Mathematics, the Conference Board of the Mathematical Sciences and the National Science Board. These groups are on record as stating that the use of calculators in mathematics instruction must be increased. This proposal is also consistent with the recommendations of a national symposium on the use of calculators in standardized testing held in September of 1986 (The College Board and the Mathematical Association of America, 1987, pp. 219-221). Permitting the use of calculators while taking tests would have two effects. First, it would lead to improved tests because it would be counterproductive to ask low level computational questions. More incisive test items which probe higher order conceptual skills would be devised. Second, it would speed a restructuring of the mathematics curriculum away from computational and towards conceptual mathematics. The tacit admission by the testing agencies that computation can be left to machines and thinking to people will allow us as teachers to do the same. Computational aids can then be introduced into all phases of our courses. We will be freed from the necessity of teaching outmoded skills simply because students will be tested on such skills in the future. This will allow us to build a curriculum which stresses understanding concepts rather than rules of computation.

If calculator use on standardized tests is permitted, the results will need to be carefully observed. When there is a sense of what the outcomes of this decision are (both intended and unintended), consideration can be given to the use of symbolic manipulators on appropriate tests.

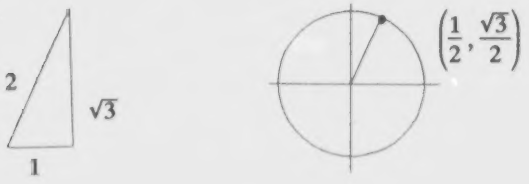
We must also consider student characteristics in any reform of mathematics education. Too often, reform efforts focus on texts and curriculum because these are

fluence on mathematics because it will allow the subject to be accessible to a wider audience. The mathematics community has received ample advance notice of symbolic manipulation packages for computers (Wilf, 1982) and calculators which feature symbolic manipulators (Nievergelt, 1987). Should we use these items in our teaching of mathematics? Of course we should! The real question is not whether they should be used, but how they can be used most effectively. What conceptual images of mathematics will students have if they use calculational devices, and will these images be satisfactory for subsequent mathematics to be built upon? Will our students miss something if the computational elements of mathematics that we have been taught and seem to prize are de-emphasized in favor of the new technology?

Consider the following example. When most mathematicians educated in precalculator days are asked to determine $\sin(\pi/3)$, they respond as indicated below.

$\sin(\pi/3)$ leads to a mental image of:

a triangle or a unit circle



which leads to $\sin(\pi/3) = \sqrt{3}/2$

On the other hand, many of today's students proceed in the following manner using a hand calculator.

$$\sin(\pi/3) = \sin(3.1415927/3) = \sin(1.0471976) = .8660254$$

So the question is: Which is the superior mental image? Is it the triangle or unit circle in the mind of the mathematician or is it the decimal result from the calculator in the mind of the student? What are the strengths and weaknesses of each? Is one image better than the other for the assimilation of subsequent mathematical knowledge? Is an amalgam of these images better? These are the questions with which we must grapple. Additionally, these are not questions which can be answered by mere opinion. There must be solid research to buttress the responses.

Conflicting Images

Some of the recent problems observed in mathematical achievement may be the result of the teachers of mathematics teaching with the triangle or unit circle in mind while the student is visualizing things with the calculator image in mind (or the functional equivalent of these concepts in other areas of mathematics). Indeed, the value system of the mathematics community which has educated most of today's teachers of mathematics has always prized "exact" radical expressions over "inexact" decimal representations, and common fractions over decimal fractions, forgetting that all are merely representations. In an age in which decimal representations are increasingly abundant because of computers and calculators, such preferences must be rethought.

When teachers use one conceptual image and students use another, we have a clear recipe for lack of understanding and even conflict. "In order to teach students what we want them to learn, we have to understand the interaction that goes on when students construct their own images of mathematics which are quite likely different than the ones we have in our minds or that we are trying to convey to them" (Steen, 1987, p. 13).

In addition to the problem of conflicting images, many students suffer from a paucity of conceptual images. A student who has only the calculator image may have too small an image base upon which to build any subsequent understanding. Several images may be needed to properly internalize a concept. In addition, there must be concern about the quality of the images. For instance, the student may have a calculator view and a trigonometric table view of the trigonometric functions. There is more than one image, but the images are not sufficiently different to facilitate conceptual understanding.

The essence of the situation is that if the calculator or computer contributes to the paucity of images by getting students to numerical answers without attendant conceptual understanding, then it will have a detrimental effect on mathematical learning. However, if it allows students more time to obtain conceptual understanding by relieving them from computational drudgery, then the effect will be positive.

Calculators and Computers

Calculators and computers will have a positive impact on mathematics instruction only if mathematics teachers seize the initiative. We must ensure that these devices are properly used by having a well-defined plan for integrating them into instruction. If we do not, students will use them regardless, and the risk of incorrect usage and the attendant possibility of incorrect concepts is a probable result.

A part of any plan must cope with the fact that in a given class different students will have different makes of machines. This forces us to teach calculator procedures individually, usually only to the few students who ask. It is my view that a calculator is as essential a component in a college mathematics class as a textbook. Given the low cost of calculators today, I see no problem in requiring students to purchase a particular calculator for a course in the same way that we require a particular textbook. The cost of a good scientific calculator is much less than the average cost of a textbook. Standardization of calculators in a class would allow for formal instruction in their proper use.

Care must be taken that the case for the positive role of computers or calculators in mathematics classes is not

the easiest things to change. If the real problem lies in other areas, neither curriculum alteration or technological innovation will help. In the absence of student motivation and appropriate use of time (Walberg, 1984; Johnson, 1987), other efforts will count for little.

The way we view students' backgrounds in mathematics must be changed. Presently, when students arrive at college, the assumption is that we need only diagnose their background deficiencies via some type of test and then "fix" these deficiencies via a remedial course. Current research indicates that this procedure is doomed to failure because students come to college not just with scattered deficiencies in mathematics, but with a total misunderstanding of the entire subject. In fact, the underprepared student in mathematics is not an anomaly; such a student is the norm (Pine, 1988). Thus, what is needed in many cases is a strategy for rebuilding students' mathematical knowledge from the ground up, rather than merely to repair some isolated deficiencies.

Conclusion

Reconceptualization in mathematics can only be accomplished if we view mathematics instruction at all levels as a system whose parts are interdependent. Elementary, secondary and postsecondary teachers of mathematics must put aside parochial interests which cause each level to blame the others for its problems. We need to look for opportunities to interact for the overall benefit of mathematics instruction.

We must renew our commitment to competent teaching. Similarly, we must insist on the use of available technology to create new conceptual images for the student. All this must be done in a way that increases the student-teacher interaction that research indicates is helpful in fostering achievement (Pascarella, 1985). The focus must be on creating many quality conceptual images upon which students can build mathematical understanding. We must open new representational windows for our students so they can form conceptual images which foster understanding rather than rote memory (Kaput, 1986). When we take such steps, we shall aid our students in seeing a whole new mathematical world.

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Challenge for the 1990s

Karl J. Smith

Santa Rosa Junior College, Santa Rosa, CA 95401

There is a crisis in mathematics education, a crisis of major proportions and the vitality of the nation is at stake. As we enter a new decade it is appropriate we reflect on the past decade and focus on emerging trends for the 1990s. Where is community college mathematics education today? Where should it be in the future? How can we get there?

Issues

- The most recent international mathematics study reported that average Japanese students exhibited higher levels of achievement than the top 5% of American students enrolled in college preparatory mathematics courses (McKnight et al., 1987). As a case in point, a Japanese semiconductor company recently opened a plant in the southeastern United States and had to use college students at the graduate level to perform statistical quality control functions; the same jobs were performed by high school graduates in Japan (Gilder, 1987).
- Nearly half of American 17-year-olds cannot perform mathematics problems normally taught in junior high school. The National Assessment of Educational Progress (NAEP) also reported that one-third of the nation's 11th-graders said they usually do not understand what their mathematics teachers are talking about in class. Among 13-year-olds, it said, 27% could not perform mathematics work normally taught in **elementary** school ("Johnny Can't Add," 1988).
- American mathematicians are becoming an endangered species. Only 362 U. S. citizens received doctorates in mathematics during the 1986-87 school year (Tynan, 1987). Those with mathematical ability are going into investment banking and computer science. It is clear that there will not be enough mathematicians to replace the baby boomers, who went to school in the 1960s, and when they begin retiring after the year 2000 we will face a crisis of major proportions. The Committee for National Study of Resources for Collegiate Mathematics (1985) concluded, "Although U.S. mathematics is still strong, many danger signals are now visible. To meet present challenges and seize current opportunities, corrective measure are urgently needed" (Madison, 1985, p. 1).
- One out of three major corporations already provides new workers with basic reading, writing, and arithmetic courses. If current demographic and economic trends continue, American businesses will hire a million new people a year who can't read, write, or count. Teaching them how, and absorbing the lost productivity while they are learning, will cost industry $25 billion a year for as long as it takes – and no one seems to know how long that will be (Kearns, 1987).
- American colleges have reported a 10 to 30 percent rise in demand over the past several years for remedial course work in mathematics for incoming freshmen (National Alliance of Business, 1987).
- Looking toward the year 2000, the fastest growing occupations require employees to have much higher mathematics, language, and reasoning capabilities than do current occupations (Hudson Institute, 1987).
- At the same time we hear these statistics we are also faced with a changing labor force. According to the U.S. Department of Labor, of the new workers entering the labor force by the year 2000, only 15% will be white men, and the rest either white women, members of minority groups, or immigrants (The Task Force on Women, Minorities, and the Handicapped in Science and Technology, 1988). But yet we continue to structure our courses and curriculum in mathematics primarily for white males.

These are some of the many issues facing education in the 1990s. I believe the most critical issues are the teacher shortage at all levels, and at the two-year college level in particular; and the retraining of faculty to face the new technologies in mathematics. Some think that the teacher shortage is due to the fact that mathematically talented people can make more money in fields other

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Karl has taught at junior and senior high schools in the Los Angeles area and was a National Teaching Fellow at Whitworth College in Spokane, Washington. He has been at Santa Rosa Junior College since 1968. He is active in local, state, and national professional groups and is currently President of the American Mathematical Association of Two-Year Colleges. Karl's main professional concern is to improve the mathematics education by being an outspoken advocate for better teaching at all levels.

than teaching. Academic salaries for mathematicians dropped 20% in real dollars from 1970 to 1985, says Donald Rung (1985) of Pennsylvania State University, who is chairperson of the AMS's committee on employment and educational policy. Salaries of full professors went from $57,000 to $48,000 in 1985 dollars. "I agree we're not poor," Rung says. "But industries are luring away prospective students before they get PhDs. Why start as an assistant professor at $21,000 when you can get more than that from industry with just a bachelor's degree?" (Kolata, 1985, p. 787) I believe a goal for the 90s must be to raise teachers salaries at all levels in order to put teaching on a par with doctors and lawyers.

It has been said that if a teacher is doing his or her job, then you can't pay them what they are worth. On the other hand, if you are working only for a salary, then any amount you are paid is too much. I'm not claiming that high salaries will solve all the problems in education, but I do think we need to start taking about $60,000 salaries for teachers at all levels. Remember good teachers cost a lot, but poor teachers cost the most.

Others believe the teacher shortage is caused by the mathematics teaching in elementary and high schools which is so poor, that many students are never really introduced to good mathematics. They never see any interesting mathematical work.

Still others put their heads in the sand and say there is not a teacher shortage. The state of Texas will never have a teacher shortage, because anyone can be given an emergency certificate to teach. All a school district has to say is that it has been unable to find qualified teachers. The state of Texas then authorizes the district to hire anyone to teach—sometimes qualified, often not. The situation is not too different in California. School districts not only can place unqualified teachers in classes, but have continued to do so because it is often cheaper. They then fool the students and parents into thinking that it is Johnny's fault he can't learn mathematics.

There are no reliable statistics on how many teachers are teaching in areas for which they are not qualified or certified. Emergency or temporary certificates are issued as a matter of course in most states. Can you imagine a state giving a temporary medical certificate to someone because it couldn't find a doctor to do the job? Or, how about the state granting emergency legal certificates? How would you like to be represented by someone who didn't go to law school but who got the job because there were no real lawyers available? How does this happen? Does the fault lie with the school districts or the states? No, it lies with us because we have allowed this to happen. We are the professionals! We let this undermining of our profession continue. It is not the students, but rather the teachers who must "stand and deliver" for our profession. For example, we must change our tenure laws to enable us to eliminate the obviously poor teachers.

Have you ever thought about the fact that what creates a teacher is a job, not a particular education or certification? This is the opposite of the way it should be. Doctors become doctors because they graduate from medical school—if they happen to be out of work, they do not cease being doctors. Same with lawyers—lawyers without clients still are lawyers. Not so with teachers. Teachers without a class are unemployed former teachers. Other professions get paid time and a half or double time for overtime work. What about teachers? We are willing to teach night school at reduced rates rather than at multiple rates.

Even worse is allowing teachers to teach in disciplines outside their area of training. The California state Academic Senate is considering minimum criteria for hiring permanent, full-time mathematics instructors. Should that requirement be a master's degree in mathematics, or should it be a master's degree in "a reasonably related discipline"? I'm reminded of the wonderful Lerner and Loew song, "If I can't be with the girl I love, I'll love the girl I'm with." The school version is, "If you can't teach the subject you know, then you'll have to know the subject you teach." Not much consolation to the physical education major who is asked to teach a mathematics class.

Still others believe the answer lies with in-service training. AMATYC, for example, has been very active in presenting summer institutes for two-year college faculty. We have an annual institute at Rick's College and this year we are adding a second institute in Vermont on the topics of Lotus and fractals. Our Academic Computer Committee is charged with fostering among the membership an understanding of the impact of computers on mathematics and mathematics education. In addition, our Equal Opportunity Committee is charged with promoting employment practices for women and minorities, and the Education Committee is looking at making recommendations concerning the minimal requirements for entrance into the two-year college teaching profession. This committee is also examining ways to enhance the professional life of two-year college mathematics faculty. I'm presently serving on a joint AMATYC-MAA committee to develop standards for certification of mathematics departments in the United States.

Solutions

Even though all of these efforts and hypotheses are noteworthy, I don't feel any of them give the solution. The answer, I believe, lies with the teacher and his or her attitude toward teaching. Some people work to live; others live to work. It is my hope that most of us strike some sort of happy balance. It is not enough to work to live—if that's all your job does for you, then perhaps you should be doing something else. Nor do I believe that living just for your work gives one the nourishment, the revitalization, that we all need. Teachers often do not talk like the winners in life; but why not? We must begin to accord ourselves more respect. We must think of ourselves as leaders. We must be willing to acknowledge our own self worth. We must be willing to say we should

be paid adequately for our skills. Love of teaching and devotion to the classroom is not mutually exclusive with expecting compensation equal to that of other professionals with our level of education. All too often I've heard love of teaching offered as an excuse for inadequate salary.

There are two kinds of persons in any organization. The shakers and caretakers; they are both important. The shakers are those people who are always looking for a better way, or a greater level of accomplishment at their jobs; the caretakers are more concerned with not ruffling the waters or disturbing a satisfactory operation. We need more shakers in education. Each of us must know we are important. Think back. Why did you become interested in mathematics? When I ask this of people the response I hear most often is that a specific teacher or course sparked interest. You must be willing to sell yourself and your profession; speak up to your administrators, you boards, and your academic senates. Making a good product and trying to sell it without advertising is like my winking at a pretty girl in the dark. I know what I'm doing, but nobody else does. We must begin to sound more like evangelists for mathematics and for teaching. We need to apply the principles that Jesse Jackson uses with his audiences when he has them repeat "I am somebody!"

A second issue is the widening gap between raising standards and competency level of entering freshmen. We see legislatures (including California, Texas, Florida, Illinois, and Maryland) requiring entrance and/or exit testing while at the same time we find students who are not prepared to do college level work. At the two-year college we are faced with rising standards but at the same time open enrollment. We find vast differences in the students in our classes. For too many of today's school population, there aren't three Rs, there are six – remedial 'ritin', remedial readin', and remedial 'rithmetic. AMATYC has just appointed a task force on Placement and Assessment to address this issue. In addition, our Technical Mathematics Committee is charged with investigating federal, state, and private funding agencies to tie together the needs of employers with what is being taught at the two-year colleges. AMATYC's Development Mathematics Committee is charged with writing guidelines of minimal competencies in arithmetic, geometry, and algebra.

Here again, I believe the solution of this problem rests with us. There are three principles which have helped me in life, and I believe in the classroom. The first is the importance of caring. Thomas Aquinas, who knew as much about education and persuasion as anyone who has ever lived, said that the only way to convert someone to your point of view is to lead him. You can't stand across the room and shout at him. You can't order him over to where you are. You must go over to his location, gently take him by the hand, and lead him to where you want him to go. You must care for him as a person, and you must not put him down or embarrass him. I have learned that you cannot fake caring. Your students will know instinctively if you are sincere. Someone can come up to you and profess their care but you know immediately they are insincere. Another person can come along and say absolutely nothing, but the way he just squeezes your elbow or touches your arm somehow says he cares. The easiest way to bridge the gap between where you want the students to end up and where they begin is to really care. The second is the importance of "thank you." Tell your students when they do a good job. Tell them thank you and offer positive reinforcement for their efforts. And the final point is to set high standards and teach the students that "opportunity always exists." Our students spend hours watching television. That most of what they see is not worth viewing is the minor crime. The major one is that they are passive. Sesame Street has been called good television, but it is passive. The constant, mindless blare trains people not to think. They are not encouraged to sit back and reflect, to mull over ideas and express them in their own minds. Mathematics requires intense concentration; television encourages nonconcentration. We must help our students learn to think for themselves and set higher (not lower) standards. But in setting those high standards we need to realize the starting location for our students. Today there are vast numbers of families where both parents work, or in which there is only one parent. Parents have less time to spend with their kids, stimulating their curiosity, answering their questions, reading to them, relaxing with them and stimulating their love of books. Allan Bloom, in *The Closing of the American Mind* (1988) asserts that families no longer think together. As a result, students don't know who Grant and Lee were, which side Hitler was on, that Los Angeles is south of San Francisco, or even the number of days in a year. Given our students' backgrounds, we must teach them to think. Although it is not an easy task, I have found that if you win your students' confidence by really caring, they will work extremely hard to reach realistically high goals. We must teach them that it is never hopeless and that opportunities for success always exists.

The National Council of Teachers of Mathematics released a major document *Curriculum and Evaluation Standards for School Mathematics* (1989). The NCTM Commission on Standards was created in 1986 and was charged with the development of a set of standards concerning the mathematics that ought to be included in a quality school mathematics program and the instructional conditions necessary for learning for that mathematics. In addition, the commission was asked to develop standards for both the evaluation of such a school program and student performance in light of the curriculum standards. I believe the implementation of these STANDARDS would alter the direction of education in the United States, and if you have not read this document I urge you to do so (National Council of Teachers of Mathematics, 1989).

Related to setting realistically high standards is an understanding of how students learn. I would like to share with you a quotation from the book *Everybody Counts:*

> Research on learning shows that most students cannot learn mathematics effectively by only listening and imitating; yet most teachers teach mathematics just this way. Research in learning shows that students actually construct their own understanding based on new experiences that enlarge the intellectual framework in which ideas can be created. Consequently, each individual's knowledge of mathematics is uniquely personal. Mathematics becomes useful to a student only when it has been developed through a personal intellectual engagement that creates new understanding. Much of the failure in school mathematics is due to a tradition of teaching that is inappropriate to the way most students learn. . . . Data from enrollments in mathematics courses reveal that roughly half the students leave the pipeline each year. . . . Mathematics must become a pump rather than a filter in the pipeline of American education. Everyone depends on the success of mathematics education; everyone is hurt when it fails. More than any other subject, mathematics filters students out of programs leading to scientific and professional careers. From high school through graduate school, the half-life of students in the mathematics pipeline is about one year; on average, we lose half the students from mathematics each year, although various requirements hold some students in class temporarily for an extra term or a year. Mathematics is the worst curriculum villain in driving students to failure in school. When mathematics acts as a filter, it not only filters students out of careers, but frequently out of school itself. Low expectations and limited opportunity to learn have helped drive dropout rates among Blacks and Hispanics *much* higher—unacceptably high for a society committed to equality of opportunity. It is vitally important for society that *all* citizens benefit equally from high-quality mathematics education (National Research Council, 1989, pp. 6-7)

I believe a third issue is lethargy in changing the curriculum. We have read much about a "lean and lively" calculus, or making calculus "a pump, not a filter," but I do not see actual change. When I move out of the circle of colleagues who attend mathematics conferences, I see apathy and lack of desire to change. At most schools, calculus continues to be taught just as it has been in the past. I do not see that most instructors use the available technologies, nor do I see any real movement in that direction. Textbook publishers are generally willing to produce only those products that perpetuate the status quo. As an author, people often say to me "why don't you do this or that in your books?" and at the same time the publishers say to me "we don't want to stray too far from the mainstream" which really means that they want a textbook that looks like all of the other textbooks. As far as I can see, none of the professional organizations or publishers are truly addressing this problem.

There is one bright spot, however. The United States mathematics community is engaged in a multistage, multiyear critical examination of its roles in research, in education, and in public policy. This ambitious undertaking, which will take another major step forward in 1990, began in 1980 when leading mathematicians became alarmed over markedly decreased flows of talent and resources into their field, and more broadly into science and technology. The focus for the analysis is not on the past, but focuses on the opportunities which mathematics provides for the future well-being of science and technology, the nation, and its individual citizens. You may have seen some of the first fruits of this endeavor. The report which came to be known as *The David Report* (1985) was published and it highlighted the flowering of mathematics and its uses since World War II and calls attention to serious signs of trouble: (1) the growing shortage of mathematicians; (2) a marked imbalance between federal support of mathematics research and support for related fields of science and engineering (David,1985). Then, in 1987 at the urging of the Joint Policy Board of Mathematics, the National Resource Council launched a comprehensive review of the college/university mathematics enterprise, analogous to the David Committee's review of the health and support of mathematics research nationally. This enterprise was called Project MS2000. I was involved in a conference held in Washington, DC in November, 1985 to review the preproposal draft of the first report of this Project 2000 committee. The first *Report to the Nation* was released last year, and has led to what has been designated *1990: A Year of Dialogue*. I hope that all of you will become involved with this dialogue which I expect will be led by your professional organizations, MAA, NCTM, and AMATYC.

I said I would offer some suggestions for the 90s on how to solve some of the problems facing our profession today. Educator's often try to solve problems by forming study groups or task forces. They are fond of making policy recommendations. But if what I say here is to make any difference, it must go beyond a policy recommendation. It must motivate you to be a mover and a shaker, and must force you to take action. The answer rests with you—not someone else. We have met the enemy, and the enemy is us! I believe this recognition is the key to solving the problem. We must be aware of the challenge, the need, and the opportunity we have before us. We can make harmonious music—but if only we heed each

other's voices – otherwise we can produce dissonance that ends up making us not part of the solution, but rather part of the problem.

Summary

In summation, here are what I see as the major problems, the goals and how we might reach those goals:

1. **The teacher shortage:** Our goal must be to expand on our present pool of people going into teaching. We need to reach out to women and minorities. We can accomplish this by strengthening (not weakening) our profession. Stand tall, and make teaching the most desirous of occupations. Pay teachers what they are worth. Consider certification (rather than credentialing) of teachers using the system not unlike the licensing of doctors, lawyers, or accountants. Revise or eliminate the tenure laws so that we can get rid of the obviously bad teachers.
2. **The gap between rising standards and lowered abilities:** Our goal must be to demand excellence of our students and not allow them to progress from one level to another without demonstrating their ability. The solution of this problem is linked to the solution of the first. We must have people competent to teach mathematics at all levels. These teachers need to excite the students with mathematical challenge, and eliminate redundancy in the classroom. We need to develop (using available technologies) reliable instruments of placement and assessment.
3. **Lethargy in changing the curriculum:** The goal is to heighten the awareness of the necessity of change among the mathematical community. This is the most difficult goal to achieve, but the publicity around the new *NCTM Standards*, the National Academy book *Everybody Counts*, and *Project 2000* are all aimed toward this goal and the year of national dialogue.

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Whither Two-Year College Mathematics?

Anthony Ralston

State University of New York at Buffalo, Amherst, NY 14260

The Forces for Change in Two-Year College Mathematics

Forever caught between the secondary school mathematics curriculum and the demands of four-year college mathematicians and scientists, reformers of the mathematics curriculum at two-year colleges must be brave souls. Still, these are propitious times for such reform for the paradoxical reason that there are so many forces for change in secondary and college mathematics that one can be assured that no clear signals will come from either the secondary schools or the colleges for several years.

Four potential forces for change in two-year college mathematics can be identified:

1. There is ferment in the secondary school mathematics community. The new standards for mathematics teaching at the elementary and secondary school levels of the National Council of Teachers of Mathematics (NCTM) were published in early 1989 (NCTM, 1989) and would, if followed, create substantial changes in what mathematics is taught in secondary schools and how it is taught. These standards have been supported by the Mathematical Sciences Education Board (MSEB) of the National Research Council which has published *Everybody Counts* (National Research Council, 1989), a much more general statement than the NCTM standards, but one whose philosophy is essentially the same. Then there is the University of Chicago School Mathematics Project (UCSMP) which has developed a six-year (7-12) secondary school curriculum with texts to match, a curriculum which is significantly different from what is normally found in American high schools.
2. Calculus reform is in the air. The proceedings of two conferences in recent years (Douglas, 1986 and Steen, 1988) provide testimony that the college mathematics community is united in believing that the teaching of calculus is generally quite poor. The National Science Foundation (NSF) has started a major new calculus initiative. All this provides a good opportunity for two-year colleges to reassess the role of calculus in their programs and how it should be taught.
3. Discrete mathematics may not have achieved the role in college mathematics hoped for by its proponents but it is very much alive (Maurer, 1989). There are a considerable number of new freshman-sophomore level books being published each year and it is beginning to make inroads into the high school curriculum. The eventual role of discrete mathematics at either the high school or college level is far from clear; what is clear, however, is that two-year colleges cannot afford to ignore it.
4. Only quite recently have a substantial number of professional mathematicians accepted that mathematics is in considerable measure an experimental science in the sense that computers and calculators can and must be the handmaidens of all mathematicians as they search for solutions to problems, for conjectures and even for the proofs of theorems. One exciting by-product of the recognition that mathematics is an experimental science is the realization that computer technology is, increasingly, a necessity in the mathematics classroom if mathematics is to be well taught (Ralston, in press-b). This stricture applies to all two-year college mathematics and provides a challenge to all two-year college mathematics teachers.

In the rest of this article I will examine each of these forces at more length. Although my remarks will be focused at the two-year college student who plans to transfer to a four-year college, much of what I shall say will also be relevant to other two-year college students.

Changes in Secondary School Mathematics

The question that two-year college mathematicians need to ask about secondary school mathematics is not whether the subject matter will change substantially but rather whether or not future secondary school graduates will be better prepared for whatever you are going to teach them than they are now. Although the previously mentioned NCTM and MSEB reports would, if their guidelines are followed, result in significant changes in the subject matter of secondary school mathematics, their most important effect would be to change the philosophy of secondary school mathematics away from developing

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Anthony Ralston is Professor of Computer Science and Mathematics at the State University of New York at Buffalo where he has been since 1965. He is a former president of the Association for Computing Machinery and the American Federation of Information Processing Societies. He has just completed a 4-year term as a member of the Mathematical Sciences Education Board of the National Academy of Sciences-National Research Council. He is the author or editor of 13 books and the author of over 50 papers.

obsolete skills in symbol manipulation and toward producing mathematical thinkers. My opinion is that measurable progress in this direction, while devoutly to be sought, is quite unlikely. The growing disaster in elementary and secondary school mathematics is likely to get substantially worse before it gets any better.

Still, two-year college mathematicians have an important role to play in stemming this tide and perhaps proving that my pessimism is misplaced. You need to make it clear to your secondary school colleagues that the students they send you need to have better mathematical abilities than they now have if they are going to succeed at your institutions or have a realistic chance of transferring to a four-year institution. It is generally true that the greatest service that college mathematicians can do for education today is not what they do in their own classrooms but rather is to send a message to the secondary schools that you insist on their doing better than they generally do now.

For this article, however, the important point is this: Nothing is likely to happen in secondary school mathematics in the next decade which will require much adaptation on your part.

Calculus and Two-Year College Mathematics

Although college mathematics educators seem generally to agree that there is a crisis in the teaching of calculus just about everywhere, there is little agreement among them on what to do about it. One of the most useful results of the discrete mathematics movement is that it has forced proponents of the supremacy of calculus to face the problems in calculus and try to do something about them. Indeed, it was the growing interest in discrete mathematics which was one of the spurs for the conference whose proceedings (Douglas, 1986) were mentioned earlier. This conference was, in fact, called by some of the attendees the counter-reformation.

What does this mean for two-year colleges? The problems of teaching calculus are at least as great at two-year colleges as at four-year institutions. Too many of your students come with poor preparation and motivation. The result is all too often calculus courses which would be better taught by mechanical engineers than mathematicians because they are little more than crank-turning courses in which students learn little else than how to churn out derivatives and integrals. This is particularly ironic at a time when symbolic systems, which can do these tasks better than virtually all students, are becoming widely available and cheaper all the time. What should two-year colleges do?

Should two-year colleges teach calculus at all? Quickly, I answer, yes, because for those of your students who wish to transfer to engineering or physical science programs at four-year schools, calculus is a necessity. So you must teach it, and however you do it, you will do it at least as well as your four-year college colleagues. But, and this is the crucial point, except for the classes of students listed above, you should probably not teach calculus to anyone else.

What good is a mechanical calculus course for all these other students when an HP-28 to say nothing of Macsyma, Maple and Mathematica can do everything you teach in such a course? Yes, calculus doesn't have to be – and shouldn't be! – a mechanical course. But how many of your students are ready for a calculus course which aims at the understanding which should be the aim of any good mathematics course rather than the skills which are the focus of almost all calculus courses? Very few, I suspect.

Is it true that, just as high schools want to teach Advanced Placement calculus because they think it adds prestige to their program, so do many two-year colleges teach calculus to students for reasons of prestige rather than because doing so is intellectually sound? The readers of this article can answer that question better than I can, but I suspect that many of you would agree that the role of calculus in many of your programs is problematic at best and that it could use reassessment and reform. One aspect of this is how you teach calculus which I'll return to in the section after next. Another aspect concerns whether or not there is more appropriate mathematics than calculus to teach many of your students. This leads me naturally to the next section on discrete mathematics.

Discrete Mathematics – The Right Mathematics for Most Two-Year Programs

Here are the essential reasons why you should wish to teach discrete mathematics to many two-year college students:

1. Except for those students planning to transfer to engineering or science programs, it will be much more useful professionally to your students than calculus. Whether they get jobs after graduating from your institutions or whether they transfer to a social science, business or arts program at a four-year college, discrete mathematics contains material of practical professional use which, frankly, calculus does not. This is especially true for any student who is going to be directly involved with computers and that certainly applies to a growing fraction of your students.
2. A good discrete mathematics course will be more difficult than a mechanical calculus course but it will not be more difficult – probably it will be easier – than a rigorous calculus course. Moreover, a discrete mathematics course deals with mathematics which is more accessible than calculus to students who have a good or fair high school mathematics background. And, it is a course in which straightforward practical applications in such areas a computer science, business, economics and biology can be given to students.
3. Some specific aspects of discrete mathematics of value at two-year colleges are:
   - Using mathematical induction, the quintessential proof technique in discrete mathematics, students can be introduced to proofs and even do proofs at a level appropriate for them.

– Related to this is the opportunity to give students a good introduction to mathematical logic which will help them develop a discipline of thought valuable in whatever they do later.

– It can be fun! Topics such as graph theory and combinatorics provide many opportunities to give students problems which are fun as well as challenging.

– It is possible to introduce topics right at the cutting edge of modern science such as chaos and fractals.

Of particular importance is the fact that in the past few years over 30 books have been published which are suitable for a two-year college discrete mathematics course (see the Appendix in Ralston, in press, a); and many more are on the way. Moreover, these books, unlike calculus texts, are not clones of each other but cover different topics, take a variety of different approaches to the subject and are written at many different levels of difficulty.

My advice is to look into discrete mathematics. You'll likely find it more fun to teach than calculus and I'm sure most of your students will find it more fun to take.

Teaching Mathematics in Two-Year Colleges

Computers have made mathematics an experimental science. Our colleagues in physics, chemistry and biology would not think of giving lectures in introductory college courses without regularly using their various experimental apparatuses in the lecture room. So, too, it now behooves mathematicians to bring computers, not just into their courses, but into their classrooms. Without exaggeration, it is fair to claim that no mathematics course taught in a two-year college can properly be taught without considerable use of computers in the classroom. Even if you don't buy "properly" in the previous sentence, you should agree that you can teach a better course with computers in the classroom than without them when your classroom is properly equipped to use computers and when well-designed software is available. According to Kraines (1989), classrooms are neither hard nor expensive to equip properly. And even if we must admit that the amount of good software available for classroom use is still much less than it might be (Cunningham, 1986), things are changing and will continue to change very rapidly in this area.

Ironically, the available software for use inside and outside of classrooms is much better for calculus (e.g. Microcalc designed by Prof. Harley Flanders of the University of Michigan) than for discrete mathematics even though teaching the latter has been mainly motivated by computers. This is because the market for calculus software is still much larger than the market for discrete mathematics software but I think you can expect the development of good discrete mathematics software over the next few years.

One motivation for using computers in the calculus classroom needs to be mentioned. Much of the calculus is about how things change, about dynamic processes. How strange then that mathematicians think it can be taught using "static" technology such as a chalkboard or an overhead projector. Unquestionably the ability to illustrate the dynamics you are lecturing about is a major reason why computers must find their way into the calculus classroom.

But I warn you, bringing the computer into the classroom will, initially at least, be hard work. Even if the software does everything you desire and even if the classroom is as well equipped as you can imagine, you will have to rethink virtually every lecture in the course. The computer in the classroom is not just an add-on which you use for periodic illustrations of the points you are lecturing about; it must be integrated into the warp and woof of your lectures. Its graphical capabilities, its symbolic capabilities, its ability to generate illustrative and provocative data – all these will require you to reassess how you teach if you are going to make proper use of the computer in the classroom.

But the basic message is clear; you **must** aim toward using computers in the classroom (and outside of it, too, of course) or you are going to shortchange your students and you will miss a wonderful opportunity to make them like mathematics more than they do now.

Conclusion

These are challenging times for all teachers of mathematics. Poor performance by American students at all levels, changes in the way mathematics is being applied, the impact of computers and computer science on mathematics – all these things create the need for major changes in **what** mathematics is taught and **how** it is taught. Two-year college teachers of mathematics have the opportunity to play a significant role in making these changes happen. You can supply a lead to your four-year college brethren who badly need it!

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The New Standards for School Mathematics and Their Implications for College Mathematics

David E. Kullman

Miami University, Oxford, OH 45056

Today we are witnessing the growth of a widespread reform movement in school mathematics (grades K - 12). This movement has its roots in the National Council of Teachers of Mathematics *Agenda for Action* (NCTM, 1980). It has been given a sense of urgency by numerous reports that show American students falling behind their European and Asian counterparts in mathematics achievement. These reports include *A Nation at Risk* (National Commission on Excellence in Education, 1983), *The Underachieving Curriculum* (McKnight et al., 1987), and *The Mathematics Report Card: Are We Measuring Up?* (Dossey et al., 1988).

Benchmark Documents

Early in 1989 three benchmark documents appeared that capture the essence of this reform. *Everybody Counts*, issued by the Mathematical Sciences Education Board of the National Research Council (1989), attempts to chart a general course for the future of mathematics education in the United States and outlines a national strategy for pursuing that course. The report describes the context in which mathematics learning occurs and addresses the nature of mathematical thought, the mathematics curriculum, methods of teaching mathematics, and ways to mobilize resources in American society to accomplish the needed reforms.

The authors of *Everybody Counts* call for a mathematics curriculum from kindergarten to college that will introduce students to the breadth and power of the mathematical sciences. All students should be taught "a common core of broadly useful mathematics," and learning should involve the students' own "processes of internal construction and experience." Calculators and computers are seen as essential tools for learning and doing mathematics.

A second document in this trilogy is *Science for All Americans*, a Phase I report of Project 2061, commissioned by the American Association for the Advancement of Science (1989). *Science for All Americans* focuses on the substance of scientific literacy by "spelling out the knowledge, skills, and attitudes all students should acquire as a consequence of their total school experience from kindergarten through high school." The chapter on mathematics provides examples of mathematical patterns that play a key role in solving problems in science, technology, and everyday life. These include numbers and arithmetic, algebraic relationships, geometric shapes, probability and statistics, and logical reasoning.

A companion report of the Project 2061 Phase I Mathematics Panel (Blackwell & Henkin, 1989) responds to the question: "What are the important ideas of mathematics that people should know and understand by the age of 18?" In attempting to answer this question, the panel considered the processes as well as the subject areas of mathematics. Its recommendations include a common core of essential mathematical knowledge and skills that will allow schools to "concentrate on teaching less and doing it better."

While *Everybody Counts* and *Science for all Americans* present broad goals for mathematics education, the National Council of Teachers of Mathematics has attempted to translate such goals into specific objectives. *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) provides detailed guidelines for the mathematics curriculum at each of three grade levels: K-4, 5-8, and 9-12. In addition, it makes recommendations for evaluating students' mathematical achievement beyond the cognitive levels of memorization and calculation.

The three grade levels share four common standards: mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematical connections. Other curriculum standards address particular content areas such as number systems, computation, algebra, geometry, functions, probability, statistics, and discrete mathematics.

The evaluation standards call for appropriate assessment methods that will not only measure a student's knowledge of mathematical concepts and procedures, but will seek information about his or her ability to solve

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David is President of the Ohio Council of Teachers of Mathematics. He earned a BA from Northwestern University and an MA from Cornell. After teaching high school mathematics and physics for two years, he returned to graduate school and received a PhD in topology from the University of Kansas in 1969. His current interests are geometry, history of mathematics, and mathematics education.

problems, reason mathematically, communicate in mathematical language, and understand the nature and power of mathematics. They also address the problem of assessing student attitudes toward mathematics.

Although *Everybody Counts*, *Science for all Americans*, and the NCTM *Standards* were developed independently of each other, there is a remarkable degree of harmony in their recommendations. All seem to agree that there is a need to reduce the emphasis on rote memorization, to involve students more actively in the learning process, to relate mathematics to other branches of knowledge, and to make better use of calculators and computers. Each report also recommends a common core of mathematics, and there is substantial agreement as to the content of such a core.

New Reforms vs. the New Math

Of course there are those who disagree with the goals proposed by the current reformers and liken this movement to the "new math" of the 1960s. It can be argued, however, that some important lessons have been learned from that earlier attempt at curriculum reform and that there are significant differences between the new *Standards* and the "new math."

In the first place, the *Standards* have a broader base of support. A draft version was circulated for nearly a year, during which it was presented at state and local meetings of mathematics teachers and school administrators for purposes of gathering feedback as well as disseminating information. Input from these meetings and from individual teachers influenced the rewriting of the *Standards* so that they are as close to representing a consensus of American mathematics educators as is likely to be seen.

A second difference can be seen in the national character of the current reform movement. Whereas the reforms of the 1960s were manifested in several curriculum projects that emanated primarily from the halls of higher education and competed with each other in various regions of the nation, the *Standards* are a product of a national organization that, through its elected and appointed leaders, is broadly representative of the mathematics teaching profession.

Still another difference lies in the recognition of the importance of applied mathematics. While the new math tended to concentrate on pure mathematics for its own sake, the *Standards* emphasize the use of mathematics as a tool in many professions and vocations. Input has been actively sought from industrial and applied mathematicians concerning basic skill areas in the mathematical sciences that are appropriate for students who will be entering the work force in the 21st century. Among the more prominent curriculum changes are suggestions for increased attention to such topics as statistics, probability, discrete mathematics, and geometric transformations. These topics are becoming more and more important in science, technology, business, and industry.

Finally, and perhaps most important, the *Standards* say as much about the spirit of teaching mathematics as they do about its content. One reason for the failure of the new math was that it was too often taught in old ways. For example, some teachers merely replaced the memorization of arithmetic tables with the memorization of truth tables. The students obediently did as they were told—often with no idea as to what it meant or why it was important.

Of course, it is possible to impose the same kind of meaningless drill on lessons in statistics, transformation geometry, or discrete mathematics. The *Standards*, however, call for an approach to mathematics learning that goes beyond memorizing facts and computational recipes. Five general goals are articulated for all students: (1) students must learn to value mathematics; (2) they must become confident in their ability to do mathematics; (3) they must become mathematical problem solvers; (4) they must learn to communicate mathematically; and (5) they must learn to reason mathematically. These goals emphasize the fact that **how** mathematics is presented is at least as important as **what** is taught.

The new *Standards* recognize the impact of computational technology in everyday life as well as in the classroom. Some of the proposed new curriculum topics have been chosen because of their importance to computer science, while some traditional topics are seen as obsolete in an era of hand-held calculators and desktop computers. It is the unequivocal position of the writers of the *Standards* that calculators and computers should be available in every mathematics classroom, and that students should be encouraged to use them—even on homework and tests.

Calculators and computers are also becoming important teaching and learning tools in mathematics. Their graphical, numerical and symbolic capabilities, combined with speed and accuracy, make it possible for students to explore various mathematical concepts at more concrete levels of reasoning. Discovery learning will become a more prominent feature of tomorrow's mathematics classroom.

The Impact on Two-Year Colleges

What impact will the new *Standards* have on the first two years of undergraduate mathematics? A quick answer might be, "Not much in the immediate future." Students currently in high school have already formed rather inflexible attitudes toward mathematics, and the influence of new content and teaching methods on them will be very modest. It may be almost the turn of the century before students who have been thoroughly inculcated with the spirit of the new *Standards* begin to find their way into collegiate classes.

In the long run, however, it is expected that the *Standards* will help mold a new breed of mathematics students. As more students take a third and fourth year of mathematics in high school, college enrollments in algebra and remedial mathematics should decline. It is hoped that

students in undergraduate mathematics classes will become more willing to explore mathematical ideas and problems, rather than merely asking for rules that tell how to get the "right answers." More students will have studied some statistics, probability, or discrete mathematics in high school, and the introductory collegiate courses in these subjects may take on a somewhat more sophisticated flavor.

The other side of this coin, however, is that students who have learned to look for more than simple answers to mathematical problems will not be satisfied with some of the beginning college mathematics courses as they are presently taught. Undergraduate mathematics instruction also needs to be infused with the spirit of the NCTM *Standards*.

The current search for a "lean and lively calculus" is an example of what can be done. Other examples include some of the innovative programs that make use of technology in the teaching of precalculus, calculus, linear algebra and statistics courses. Although college students may be older, they are often no more mature mathematically than their counterparts in the high schools. College faculty share common educational goals with elementary and secondary school teachers of mathematics. Teachers at all levels can learn from each other. The collective result of such collaboration must be to make mathematics more interesting and meaningful to students.

Mathematics courses for elementary and secondary teachers must also undergo significant changes. The next generation of teachers will not only need instruction in topics not traditionally found in school mathematics, but they will also need to be exposed to pedagogical models that illuminate mathematics as problem solving, as a means of communication, as a means of deductive reasoning, and as a subject with many connections to other disciplines.

Summary

While the reports cited in this article are capturing national attention, they should not be taken as a sign that solutions to the problems of mathematics education have been found. In fact, the writers of the reports make it clear that their recommendations are only a beginning. There are no quick fixes to present educational problems, and the reform process must extend well into the 21st century.

The thought of what effects this reform movement may have is exhilarating. Only once in a lifetime are we likely to have such a golden opportunity to influence the future of mathematics education. It is also a sobering thought. If we fail to achieve genuine reform, another generation of students will be condemned to innumeracy, and our nation's ability to compete successfully in a world that is becoming ever more dependent on mathematics will be further eroded. We can make a difference. We must make the difference!

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Mathematics and the Naval Officer

Richard F. Maruszewski

United States Naval Academy, Annapolis, MD 20102

Franklin D. Roosevelt once said, "Peace, like war, can succeed only where there is a will to enforce it, and where there is available power to enforce it." The United States Naval Academy is proud of its contribution to the enforcement of peace through its preparation of an able body of officers for entry into the Navy and the Marine Corps. The mission given the Academy is very wide because these officers may enter the surface Navy, the submarine force, Navy or Marine air, the Marine Corps, as well as a myriad of other specialties after graduation. No matter what the choice, however, the new officer will be entering an exciting, rapidly changing, highly technological world for which his or her Academy education has laid the proper foundation.

If flying is selected, as detailed in Hessman and Hessman (1985, August), the officer may encounter a trainer called TACTS (Tactical Aircrew Combat Training System). TACTS is one hundred miles long and fifty miles wide and is located in the Naval Air Station in Fallon, Nevada. It is a marriage of modern computers and microelectronics which is able to keep track of 36 aircraft and 50 weapons simultaneously. A trainee flying over this huge training simulator is tested under a wide variety of circumstances including "dogfights" with "foreign aircraft."

If our young officer chooses the surface Navy, as noted in a later article by Hessman and Hessman (1985, September), he or she may become part of a LAMPS antisubmarine warfare team. This team consists of a helicopter, a surface ship such as a destroyer or a frigate, and an avionics/computer system and subsystems. In a typical mission, when the presence of an enemy submarine is suspected in a certain sector, the helicopter is sent out to the target area and ejects up to 25 soundbuoys of various types. The soundbuoy readings are radioed back to the helicopter, analyzed and coded, and then transmitted back to the parent ship for instantaneous processing and analysis. The mathematics involved is almost as complicated as that for a moon shot. The LAMPS mother ship is moving at a high rate of speed, changing course constantly. The helicopter is moving up, down, and sideways at speeds approaching 100 knots. The target submarine is moving at over 30 knots, constantly changing course and speed to avoid detection. The computers provide immediate solutions to this constantly changing problem, not only identifying the submarine and its present location but also projecting its future track precisely.

If the nuclear Navy is chosen, either surface or submarines, our young officer begins by attending Nuclear Power School (see Southard, 1985). This school is a rigorous six month introduction to naval nuclear power. Then comes another six months at one of the three nuclear prototype training units, which are essentially the engineering plant of a ship or submarine mounted on land. Thus, in a single year, the young officer is taught the theory and practice of operating a naval nuclear reactor.

These three examples are typical of the technological world that our officer is entering. Obviously, a proper preparation in mathematics and the sciences is necessary for the officer to understand and function in this environment. To accomplish this, each midshipman at the Naval Academy is given a solid base of such courses. For example, if a midshipman is an English major, this same midshipman will still take two semesters of chemistry, two semesters of calculus based physics, one semester of electricity, one semester of electronics, two semesters of naval engineering, and two semesters of weapons systems; all buoyed by a mathematical foundation of Calculus I, II, and III and Differential Equations. Other advanced mathematics courses such as Matrix Theory, Probability, and Statistics are similarly valuable and are popular electives. In fact, our last graduating class took, on the average, over 19 hours of calculus or above mathematics.

It is also quite obvious that the technological environment that we have alluded to is not static, but rather rapidly changing. Therefore, our midshipman must also learn "how to learn." There is no better major to accomplish this goal than a mathematics major. As a USNA mathematics department pamphlet edited by Andre (1984) states, a person with a background in mathematics is prepared to solve problems in a wide variety of fields. One reason is that mathematics teaches you to

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Richard F. Maruszewski, Jr. received his BS and MS in Mathematics from Marquette University. He received his PhD from the University of Wisconsin in Milwaukee. He is currently a member of the Mathematics Department at the United States Naval Academy after having spent 12 years at Jamestown Community College in New York. While at Marquette he was a member of the NROTC and received his Navy Commission upon graduation. He is a Captain in the Naval Reserves and is the Commanding Officer of a reserve unit in Baltimore, MD.

think logically. The capacity for critical thinking is highly prized. The mathematics major also teaches its students to read material carefully, to analyze problems, and to communicate solutions in a precise manner. These skills are invaluable in dealing with future technical problems, giving our major the capability to attack a new problem in a careful and disciplined manner.

Also, mathematics plays a central role in virtually every technical or scientific field, every type of engineering, and throughout the social sciences. Current research in physics, for example, requires almost as much background in mathematics as in physics. Similar statements can be made about economics, computer science, electrical engineering, and many other disciplines. As a consequence of their undergraduate training as mathematics majors, recent USNA graduates have received master and doctoral degrees in mathematics, operations research, computer science, physics, astronomy, acoustics, systems engineering, nuclear engineering and economics.

The success that mathematic majors have had in such diverse fields is not surprising. As we have indicated above, the mathematics major nurtures logical thinking, precise expression, defining and solving problems, and the ability to reason abstractly and to generalize. These skills are invaluable in a submarine, a jet fighter, a court of law, a major corporation, and practically anywhere else!

CLOSING NOTE TO THE READER: The Naval Academy encourages applications from graduates of two year colleges. They must go through the same nomination process as any other prospective midshipman. Also, they must spend four years at the Naval Academy. However, they will be given advanced standing based on validation exams, and, as a result, it might be possible for them to participate in Masters Degree programs at nearby graduate schools during their senior (first class) year and the following semester. The author will be glad to answer questions and assist as needed at (301) 267-3604.

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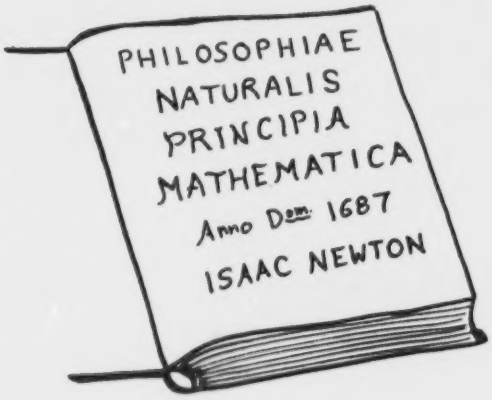

- A Humanistic Calculus – by William E. Rosenthal

  "The effort to offer a humanistically based course in the calculus originally arose from the long-standing desire to teach the concepts of the calculus to any intellectually curious layperson by locating the subject within an historical, philosophical, and foundational perspective."

- A Discrete Mathematical Modeling Course for Liberal Arts Students – by Sheldon P. Gordon

  "The emphasis in the course is on demonstrating that mathematics is not just a collection of manipulative skills developed for future courses nor is it a collection of abstract results studied for their own sake. Rather, the course demonstrates that mathematics is a vital dynamic discipline that allows us to address real problems...."

- Integrated Precalculus and Differential Calculus: A Lagrangian Approach – by Francesca Schremmer and Alain Schremmer

  "Given the problems encountered by our students, it seems most important that the degree of difficulty, as perceived by them, not increase exponentially as the course progresses. This is precisely what happens with the conventional precalculus – calculus architecture...."

- The Introductory Statistics Course of the 1990s – by Mary R. Parker

  "The challenge to us as we create the introductory statistics course of the 1990s, is to focus on the aspects of statistical literacy that will be most useful for interpreting statistical analyses. This includes determination of whether the data are adequate to answer the questions of interest, whether the techniques are appropriate for the data and questions of interest, and whether the results of the applications of the techniques are interpreted correctly."

- A Fractal Outline of a Fractal Course – by Avery Solomon

  "A fractal course could have several orientations: toward the liberal arts, toward mathematics or science majors prior to calculus, or towards those with a one-semester or year calculus background."

A Humanistic Calculus

William E. Rosenthal

Ursinus College, Collegeville, PA 19426

"If it's broke, then fix it."

Anonymous

As of now, mathematics education in the United States is broken of spirit as well as bereft of tangible outcome. This paper is not an attempt to create yet another position paper that bemoans the sorry state of our profession, but is meant to introduce and elaborate upon a radical project that arises from my realization that ten years of complaining about poor student preparation, poor student skills, and poor students had not enriched these students' lives in any way. The centerpiece of this impetus is a course entitled "Humanistic Calculus," which is currently undergoing its initial offering at Ursinus College.

The effort to offer a humanistically based calculus course arose from a long-standing desire to teach the concepts of the subject to any intellectually curious layperson by locating it within an historical, philosophical, and foundational perspective. The course is primarily pitched at persons who, largely due to learned helplessness generated outside of their control, are not likely to benefit from further instruction that reminds them of past mathematics classes. It also offers new insights on familiar material to the more mathematically accomplished student.

I chose to implement this idea in a nonsymbolic, noncomputational, and vernacular-based manner by presenting and discussing the subject almost entirely in the English language. My first stab at a humanistic calculus attempts to connect the discipline with the mainstream of human thought and gives primacy to a verbal characterization of the *limit*. This and other **content** areas are well established early in the course's development, but it was not until recently that the pedagogical **process** began to emerge and take shape. Both content and process will be treated in detail a little later on, but first I would like briefly to discuss the gestation of both the general notion of a humanistic calculus and the specific implementation that has evolved.

Bill Rosenthal, Assistant Professor of Mathematics and Computer Science at Ursinus College, earned a PhD in mathematics from the State University of New York at Stony Brook in 1983. His current focus is helping students to enable themselves to be active learners, but his professional interests also include combinatorics and numerical analysis. Much of his off-campus time and energy is devoted to correcting his own illiberal education.

Rationale, Overview, and History

The seed that has grown into the present project was planted approximately five years ago; this was the realization that conventional calculus pedagogy was indeed broken and that unconventional means would be required to fix it. To this day, our schools continue to produce generations of students with substantial computational ability yet little idea of the **meanings** of the quantities being computed. Curriculum planners and instructors often lose sight of the fact that the **what** and **why** of a subject should at worst accompany and preferably precede its **how**. The standard curriculum makes a weak if well-intentioned attempt to adhere to this principle, but empirical evidence indicates that many calculus students leave their courses without a significant appreciation of the subject. (It is heartening to see that many of the curricula presently being proposed for "Calculus for a New Century" are based upon redressing this imbalance.)

Thus was borne the desire to teach the calculus in a novel fashion. The hypothesis that the calculus could indeed be taught in the manner described was tested successfully at Hartwick College in June of 1985, when a one-week course entitled "Calculus Without Formulas" was offered to six Hartwick faculty members, most of whom had not previously been exposed to the subject. Each participant found his mathematical appreciation enriched and renewed; none felt that the absence of computational algorithms detracted from his experience. This outcome led directly to the course currently being offered.

Although I am unaware of any other ongoing initiative that uses the methodology of the humanities (and social sciences) as a basis for calculus instruction, many authors have successfully incorporated humanistic aspects in works suitable for the undergraduate classroom. The tone and spirit of W. M. Priestley's beautiful book (1979) is strikingly similar to the perspective described in this paper. Sawyer's plain-language paperback (1961) offers

an introduction to differentiation that can be read by an engaged and concerned layperson on her own.

Acknowledgment should also be made of the profound influence of Belenky, Clinchy, Goldberger, & Torule (1986) on the development of Humanistic Calculus. In particular, the related necessities of enabling students to construct their own knowledge and responding to students' voices were made clear through the authors' poignant case studies.

Many high school graduates do not appear to understand the meanings of the mathematical symbols that they have been taught to manipulate. Since persons cannot be expected to learn through discourse in a language that they cannot speak, I chose to present an English-language based version of the calculus. Symbols are used freely and appropriately, but are treated as **words** that are linguistic creations, and as such are no different in character from words in English or any other language. The purpose here is not to pander to mathematical illiteracy but merely to **respond** to it, and the hope is that the student who is not yet fluent in mathematical dialects, and in all likelihood never will be, will by this approach be empowered to achieve a meaningful appreciation of one significant aspect of the discipline.

Course Description: Content

Curious laypersons and colleagues alike have inquired if Humanistic Calculus is to be thought of as a presumably easy "Calculus for Poets." The answer is no. Participants are clearly informed, on the syllabus and also during the first five minutes of the first class, that they have not enrolled in a "gut." It is difficult to conceive that any course that involves extensive writing about mathematical topics, expects students to **question** as well as to answer, and is centered on the notion of limit could be perceived as such.

The limit is what demarcates the calculus from precalculus mathematics, and although a student may learn **about** calculus without an understanding of this notion, there is no learning the calculus without appreciating the limit. My feeling is that the failure of the present curriculum can largely be traced to the attempt to base a body of knowledge on a vague and ill-defined foundational concept. Imagine the study of chemistry without a full understanding of the meaning of a molecule or an assay on English composition prior to the knowledge of subject-verb agreement! Moreover, the standard treatment of limits so confounds most students that they are left unnerved, unglued, and unprepared for the sequel.

The Humanistic Calculus syllabus treats **only** the limit of a sequence. There are two compelling reasons for this choice. One is that it is historically faithful. The first calculus problem (i.e., the first problem involving limits) to be solved was the finding of the area inside a circle, whose solution employs sequences of inscribed and circumscribed regular polygons. It is natural and effective to start a course in the calculus with the start of the calculus, and a subsidiary advantage is that the first content area in an intimidating subject is one that is already familiar to students.

The second reason for limiting the dialogue to limits of sequences is less lofty: Limits of sequences are easier to apprehend than limits of functions. This approach causes no difficulties in the treatment of integration, but an atypical definition of derivative is required. I chose to use the result that if $\lim_{x \to a} f(x)$ exists, then the value of the limit is $\lim_{n \to \infty} f(x_n)$, where $\{x_n\}$ is any sequence that converges to a. This produces a (verbal) definition tantamount to

$$f'(t_0) = \lim_{n \to \infty} [(f(t_n) - f(t_0))/(t_n - t_0)],$$

where $\{t_n\}$ is any sequence converging to t_0; **implicit** here is the assumption that f is differentiable at t_0. Also, since the derivative is introduced via the problem of finding instantaneous velocity, only strictly increasing sequences converging to t_0 are considered. Reductions such as these are characteristic of the style of the course.

The limit of a sequence is given its standard, precise definition (the "ε-N" definition), however. An English-language description results from an exhaustive analysis of the properties of the sequence of regular polygons inscribed within a circle. The usual objects of study—the integral and derivative—can then be **constructed**, as answers to "natural" questions, by the limiting process. Silently assuming its existence, the real number system is also constructed "from scratch," commencing from the Pythagorean perspective that positive real numbers correspond to lengths of line segments. In particular, the employment of an ellipsis in writing irrational numbers is exposed as an abuse of language that disguises the role of a limiting process.

Historical aspects are both woven into the fiber of the course's content and proudly worn on its sleeve. The contributions of dozens of persons are offered, with Newton and Leibniz viewed as "midwives" who delivered the calculus after a 2000-year gestation rather than as its **only** two creators. Many discussions involve the symbiotic, yet often tempestuous, relations of the calculus with other areas of human experience.

Conspicuous by their absence are the standard algorithmic rules of the calculus. A principal motivation in forming the creation of "Calculus Without Formulas" was the desire to teach a course that could properly be called Calculus but would not contain

$$d(x^n)/dx = nx^{n-1}.$$

Both the prototype course and its successor are faithful to this goal, whose purpose is best stated by Sherman Stein (1987a; 1987b) as a corollary to Gresham's Law: "Algorithm drives out thought."

Course Description: Process

The methods of Humanistic Calculus form a melange drawn from a wide variety of techniques commonly employed in courses in the humanities and social scien-

ces. It is ironic that students are much more familiar with these methods than the instructor, although all participants are not fully comfortable with their use in a mathematics classroom.

First and foremost, there are no lectures. Class meetings are conducted as semi-Socrative dialogues. I serve merely to choreograph the discussions, occasionally tossing in clarifications of and elaborations upon student comments. This style of discourse is made possible by the textual material, which consists of lecture notes, each section of which is designed to be processed prior to being discussed in class.

A striking feature of the notes is their first-person and conversational style. Motivated by the belief that learning is facilitated when people write as people speak, the discussion was made to scrap every suggestion of standard academic patois and attempt to talk **with** rather than **at** the reader. Exercises appear on almost every one of the 340 pages. These ask the student to interpret, extend, synthesize, discover, fill in, demonstrate, relate, connect, sense, review, sketch, analyze, contemplate, induce, consider, investigate, create, react, deduce, and (occasionally) to compute. Many of these bring the participant to the point where he can continue the exposition of the text, which subsequently picks up with my own response to the exercise. Although occasionally awkward, this technique seems to have helped create the sense of a dialogue.

Much of the humanism of Humanistic Calculus derives from the iconoclastic perspective that mathematical knowledge is **subjective.** Objects and processes are perceived as created by human choice. For example, the incorporation of a regular partition, as opposed to some other partition, in the building of a Riemann sum, is viewed as one choice drawn from among many viable alternatives. Notation in general, and Hindu-Arabic numerals in specific, are presented as humanly chosen names.

The point of view that mathematics is subjectively created by human choice is emphasized throughout the notes and discussions in the hope of undoing a pervasive fixation on "the answer." Many students report that they have been paralyzed in mathematics classes by the perception that since there is one way to get one answer, the only manner in which problems may be solved is to manufacture **someone else's** solution. (How many times have you heard "How did **they** get the answer?") A subjective and creation-oriented viewpoint enables students themselves to be creators; it empowers each participant by finding validity in her own answer, whatever the nature of this response may be.

Students are encouraged, in the strongest possible way, to be active learners. Class participation is determined by **whether** or not a student speaks, not on **what** he says. The lecture notes are written so as to lead participants to discover on their own. As a third means for students actively to engage with the material, writing complements both reading and speaking. Each person maintains a journal and is asked to write three essays as part of the course requirements. Finally, every student is requested to keep a triweekly individual appointment with me.

Practicalities of a Humanistic Calculus

There is no avoiding the fact that the creation of this course requires an exceptional commitment of individual, departmental, and institutional energy. The preparation alone is daunting. A prospective instructor must be steeped in the history and philosophy of the calculus. Background readings in learning style and cognitive theory (e.g., Belenky, Clinchy, Goldberger, & Torule, 1986) are useful. Extensive writing is also needed, whether or not book-length lecture notes are to be used.

A department must first accept the premise of this experimental course. Once allowed to join an already overcrowded curriculum, it should be offered in at least two consecutive years. Preferably, two or more persons should be prepared to teach Humanistic Calculus. Responsibility for informing the college community (administrators, academic advisors, prospective participants) about something new and unfamiliar also falls squarely upon department members.

Support at the institutional level is also crucial. The unusual amount of time and energy required to prepare for, create, and teach a humanistically based calculus course makes almost mandatory the availability of release time and internal grants. The faculty and administration must be willing to allow the course to satisfy any mathematics requirement that is in place, for otherwise there will be very few takers for an offering with the word "calculus" in its title.

In regard to the syllabus, appropriate questions have been raised by concerned and respected colleagues as to whether the project described in this paper falls under the rubric of mathematics. Each individual's response is a function of her subjective answer to the question "What is mathematics"? It is worth remembering here that the ancient Greeks did not divorce mathematics and philosophy, and that there was no need for the term "Renaissance man" until after the Renaissance. Given the sorry experience of most students in post-junior high school mathematics, I am comfortable that Humanistic Calculus is faithful to the physician's credo: "First, do no harm."

Relevance to Two-Year Colleges

The liberal-arts orientation of Humanistic Calculus is likely to be clear; in fact, the course's original title is "Calculus for the Liberal Arts." Since the offering is for the benefit of liberal-arts students, at a liberal-arts college, and by a mathematician with a liberal-arts bent, it is natural to questions whether it has a proper place in a two-year college's curriculum. This paper is written in the firm belief that the answer is **yes.** Since nothing is

assumed beyond the eighth-grade level, most two-year college students are sufficiently prepared.

The vocational interests of many enrollees at community and junior colleges may seem to render the humanistic perspective inappropriate for such persons, who typically take mathematics courses designed to reinforce and bolster their skills. Yet, since it appears as if remediation often does little more than afford students the opportunity not to learn in college what has not been learned in high school, Humanistic Calculus can provide a genuine alternative to the continuance of a legacy of failure. With the help of a supportive instructor, participants can be unburdened of the aversive conditioning that accompanies them into the mathematics classroom. The stigma of retaking secondary school mathematics is replaced by the pride that accompanies knowledge of an advanced subject thought of as unknowable – if thought of at all.

There are some special practical difficulties involved in creating a humanistically based calculus at the two-year college level, principal among them the fact that four-year colleges will be disinclined to grant transfer credit for a course for which they have no apparent equivalent. Yet many such institutions offer a lower-level survey of mathematics, and I feel that a reading of the Humanistic Calculus syllabus would convince departments that the course described here is an appropriate equivalent.

The Outcome to Date

The premiere of Humanistic Calculus is half-complete at this writing. What follows are some perceptions of a process still in progress, many of these transcribed from students' voices.

Participants have taken well to the viewpoint that mathematics is humanly created. They have embraced the opportunity to recreate constructions of formulae that have been previously handed to them as quasi-commandments. Their comments indicate widespread and buried frustration and anger with an instructional style that treats knowledge as revealed truth.

Although aware that Humanistic Calculus is "different," students cannot spontaneously erase the counter-pedagogical script that directs many of their attitudes and activities. Some still ask "What will be on the exam?" when questions are called for; many continue to be silent in spite of continual reminders that they are to be rewarded for the mere act of speaking; and their attendance is as irregular as in more conventional courses.

An unexpected irony is that the vernacular-based approach, intended to promote learning in the student's own language, is itself a two-sided coin. The lecture notes are effective only when read critically and actively, and many students are indeed able to employ them in the personal creation of knowledge. Yet others lack sufficient ability to advance beyond the literal meanings of the printed words or even to understand the denotations of the terms used. And some students have simply , for whatever reasons, not "taken" to the course.

There have been some unmistakably positive reactions. Students have spoken of "A-ha!" experiences and also syntheses of knowledge previously absent from their mathematical education. A number have spoken in joyous terms uncharacteristic of typical self-descriptions of learning. For the most part, participants appear to enjoy the reunion of mathematics with other areas of human thought and also to welcome references to philosophy, history, and literature.

The strong positive correlation between perceived activity and performance is notable. Almost without exception, those students who did well on the midterm examination are those who seem to have opened themselves to a new way of knowing and learning. This outcome corroborates the risky promise that not only will effort count in and of itself, but that effectively directed effort will be the principal creator and mediator of knowledge.

Conclusions and future Prospects

It is foolhardy of me to attempt to offer objective conclusions about an ongoing project that has occupied four years of my life. You are encouraged to react for yourself, informed by a few plain and plainly stated subjective reactions. I also invite you to contact me for further discussion.

I have learned much of value. This reeducation in the calculus and the consequent broadened perspective now nurture the entire body of my teaching. More important, all indications are that participants in Humanistic Calculus will learn much of value. Each will take away some material knowledge and will grow as a learner.

Perhaps you share with my colleagues some doubt as to whether Humanistic Calculus may properly be called mathematical. But if the claims in the previous paragraph are given credence, then I expect that at the very least, you will concur with my perception that this unconventional course is genuinely **educational.**

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A Discrete Mathematical Modeling Course for Liberal Arts Students

Sheldon P. Gordon

Suffolk Community College, Selden, NY 11784

For the last three years, Suffolk Community College has offered an Honors Program which involves a variety of discipline-oriented and interdisciplinary courses designed to challenge and stimulate students who have high ability and high levels of personal motivation. Typically, the students in this program are non-traditional students. They are primarily mature women who are returning to school for the intellectual stimulation or for preparation for careers. Unfortunately, few of these individuals have strong traditional math backgrounds and so most of the obvious mathematical offerings that would be considered honors level are inappropriate for them. As a result, while the overall Honors Program has been widely acclaimed as a major success, the offerings in mathematics have left much to be desired.

At the same time, one of the most significant developments in mathematical education over the last decade has been a new emphasis on contemporary applications of mathematics. To date, this emphasis has manifested itself primarily as a course in mathematical modeling for junior and senior math majors at many four-year colleges and universities. It essentially brings together various strands of undergraduate mathematics such as calculus, differential equations, linear algebra, statistics, probability, and numerical methods and utilizes them to model a variety of real-world situations.

Under a grant from the State University of New York's program for Improvements in Undergraduate Instruction, the author has developed a comparable course in mathematical applications at a considerably lower mathematical level which is appropriate for students in Suffolk's Honors Program. Rather than building on calculus-based mathematics, the underlying mathematical structure used is discrete mathematics. In particular, most of the models used are based on difference equations (instead of differential equations), as well as some simple matrix algebra, some notions of probability, graph and network theory and geometry. All of these topics and their applications are easily accessible to students whose mathematics background includes only intermediate algebra, but who are bright and well-motivated.

The emphasis in the course is on demonstrating that mathematics is not just a collection of manipulative skills developed for future courses nor is it a collection of abstract results studied for their own sake. Rather, the course demonstrates that mathematics is a vital, dynamic discipline that allows us to address real problems in a wide variety of interesting and essential areas of modern life such as biology, psychology, political science, and economics.

Moreover, the course is designed to involve the students directly in developing and exploring the mathematics through the use of sophisticated computer graphics programs (developed by the author) which implement various models. In addition, the students are expected to conduct their own individual projects to apply the various models to actual real-life circumstances.

Content of a Discrete Mathematical Modeling Course

The topics covered in this one semester course include:

A. Difference equations and their applications in areas such as biology, technology, and political sciences, including: 1) the exponential growth model; 2) the logistic (inhibited growth) model for populations; 3) the spread of technological innovations (such as the use of home computers, color TVs, VCRs, and CD players); 4) the spread of disease and epidemics; 5) the learning model in psychology; 6) the Fibonacci sequence and its applications in population growth (Newton's rabbit model, plants, flower and seashell development), as well as the Golden Ratio and its applications in art, psychology and marketing; 7) population models for two interacting species (the discrete Lotka-Volterra predator-prey model and others); 8) Richardson's arms race model for competing nations.

Sheldon Gordon received his PhD in applied mathematics from McGill University. He currently devotes most of his attention to applications of computers throughout the entire mathematics curriculum, including calculus, differential equations, modeling and statistics.

B. Matrix algebra and its applications including: 1) discrete optimization via the linear programming model (problems such as the resource allocation, the diet, the transportation and the assignment problems); 2) discrete optimization via the integer programming model (such as the traveling salesman and the knapsack problems); 3) Leslie's model for population growth.

C. Graph and network theory and their applications, including: optimal routes for mail delivery and for garbage pick-up.

D. Elementary probability theory and its applications, including: 1) waiting time predictions for drivers at red lights; 2) waiting time simulations for customers in line in supermarkets, banks, or fast food restaurants; 3) opinion polls and surveys; 4) ESP tests.

E. Introductory data analysis and its applications, including: 1) simple regression models; 2) interpolation models; 3) applications involving sports data, population data, sales data for products or businesses, stock data, scientific measurements, etc., all of which are readily available from newspapers and standard library references.

F. Geometric models including: 1) properties of conic sections (i.e., reflection property of the parabola or ellipse); 2) geometric patterns in mathematical curves such as hypocycloids and epicycloids; 3) geometric patterns arising from the mathematical theory of chaos.

We note that some of these topics (linear programming, graphs and networks and some probability) are already covered in standard courses in finite mathematics. However, such courses tend to be geared to business students with emphasis on business applications and are not particularly appropriate for a liberal arts audience. Furthermore, this discrete math modeling course actually pulls together many of the strands (sequences, matrix algebra and probability as well as the traditional topics of algebra and geometry) that have been introduced into the modern high school mathematics curriculum or the two-year college math curriculum by reinforcing them and demonstrating their applicability.

In developing this course, the author prepared a comprehensive set of lecture notes and accompanying software for the IBM PC computer covering most of the topics listed in A, D, E and F. Materials used for most of the other topics in B and C are found in existing commercial texts and software packages.

Course Methods

The approach used in the course involves:

1. introducing and developing a mathematical topic (such as first order difference equations and their solutions);
2. introducing a problem or situation;
3. identifying the mathematical factors and modeling them via an appropriate equation;
4. experimenting with the model on a qualitative basis via computer exploration, whenever possible; and
5. having the students conduct individual research activities based on a number of the models studied.

One key feature to this course is the extensive use of computers. The author's programs are used in class to demonstrate the results of implementing the various models both numerically and graphically. They are also used individually by the students in a computer laboratory setting. Primarily, the problems are used to experiment with the effects of changing various parameters and/or initial values on the quantity under investigation. They help the students to visualize the behavior of solutions in a qualitative way.

For instance, with the predator-prey model, the students are able to alter any of the four population growth parameters and/or the initial population of the two species to see the qualitative effects on the population over future time. In turn, they come to appreciate the interactive nature of the two populations and their effect on one another. Thus, the mathematics is transformed into an exploratory activity with the computer doing all the difficult work while the students are free to think about the significance of the concepts.

Further, the course provides an outstanding opportunity for involving students in personally using mathematics. This involvement is accomplished through a series of individualized student "research" projects which relate many of the various models studied to actual real-world data. As examples, a few of the assigned projects are:

a) ***A waiting time model for cars at a red light:*** Students are assigned the project of selecting an intersection of their choice, measuring the characteristics of the light cycle, using the general mathematical model developed in class (Gordon & Gordon, in press) to predict an expected or mean waiting time at this particular intersection, selecting a random sample of cars which pass through the intersection, measuring the waiting time for each, and comparing the mean of the observed data to the predicted data.

b) ***The spread of technological innovations:*** Students have to collect data (from a library) on the annual sales of any device of interest (say VCRs) over the last ten years and use an appropriate computer program to match the actual data to the predictions based on the discretely logistic model by experimentally selecting the most appropriate parameters.

c) ***The analysis of data:*** Students select some set of data (say the prices of a stock of interest to them) and use the computer program on interpolation to produce an interpolating polynomial which matches the prices of that stock.

An Example

To illustrate the type of approach used in the course, suppose that the discrete exponential growth model based on the difference equation

$$\Delta x_n = (x_{n+1} - x_n) = ax_n$$

has been studied. We point out to the students that it is unreasonable to expect that any real process can continue to grow exponentially without some additional (inhibiting) factors coming into play to slow down the growth. One way to account for this mathematically is to alter the above difference equation by introducing an extra term which acts to retard the rate of growth and so we would then consider

$$\Delta x_n = ax_n - bx_n^2$$

for some new constant b. Using a computer program which allows the user to explore graphically the effects of different values for a and b on the population, students see that an appropriate shape (one where there is exponential growth initially flowed by a leveling off as time goes by) occurs when b is considerably smaller than a. In turn, students investigate the level of the equilibrium (known as the carrying capacity of the environment). Further, the computer program allows students to explore the effects of using different initial populations, both below and above the limiting equilibrium value.

Once this logistic model has been introduced and studied, students are able to conduct individual projects based on it. The spread of many technological innovations follows this model. For example, students may be asked to collect data on the annual sales of some products over the last 10 years, graph the data by hand to assure them that the growth pattern agrees with the logistic model, and then use a second program that will allow them to match their data to the predicted model by calculating the best values for the coefficients a and b.

Finally, each student is expected to prepare a formal project report describing the results of each of these investigations. In this way, the students become active participants in using the mathematics rather than just passive observers in the classroom.

Conclusions

In general, the discrete mathematical modeling course provides the honors students the opportunity to apply mathematics to a wide variety of areas. As a result, it is hoped that the experience will kindle a deeper appreciation for mathematics and its uses. It also provides the students with the opportunity to gain an appreciation for the power and ease of use of computers as investigatory tools. Further, the individual projects serve to involve the students in mathematical developments and simultaneously provides them the opportunity to develop their writing skills. Moreover, such activities are particularly well suited to the intended audience – the highly motivated students in the College's Honors Program. Finally, the author expects that many of these students are likely to be turned on to mathematics, in particular, and more technically oriented subjects, in general, as a consequence of this exposure to mathematics.

A further advantage for a course such as this is that it can be easily modified to reflect many other choices of mathematical models. Consequently, it is likely to evolve over the coming years, especially as other instructors become involved and introduce some of their own favorite applications.

In addition, the possibility also exists of developing a slightly less sophisticated version of this course as a part of the math department's normal offerings. (In this context, "less sophisticated" refers more to the level of motivation of the students than to the level of the mathematics covered.) At Suffolk as well as at schools throughout the nation, the traditional Liberal Arts Survey of Mathematics course is dying out as students demand more immediate relevance and applicability in their courses. This is certainly in line with one of the major recommendations of the prestigious conference on *New Direction in Two Year College Mathematics* (Albers, Rodi & Watkins, 1985) that all mathematics taught be problem driven rather than introduced for its own sake or some eventual need in a subsequent course.

Thus, in place of the Survey of Mathematics course, we see far greater enrollment in courses such as introductory statistics and finite mathematics. From this point of view, a relatively low level course in contemporary mathematical applications based on discrete mathematical models has the potential to develop into a major offering at many institutions.

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Integrated Precalculus and Differential Calculus: A Lagrangian Approach

Francesca Schremmer
West Chester University
West Chester, PA 19383

and

Alain Schremmer
Community College of Philadelphia
Philadelphia, PA 19130

There is surely no need to make a case for the importance of the differential calculus, the mathematics of change, for the technological future of this country. On the other hand, the widespread dissatisfaction with calculus instruction does not seem to have resulted in much change. In fact, there is little agreement about the cause of the "crisis".

We do not believe that the calculus "crisis" is the result of the *methods* and *techniques* used in teaching the course. Nor do we believe that it is the result of the students' lack of motivation. We believe that the 50% nationwide failure rate in first year calculus is mainly due to it having been *made* inherently difficult. To see how the calculus was presented until about the thirties, see J. W. Young and F. M. Morgan (1922). But starting then, the mathematical community insisted on a conception of the calculus based on limit considerations that is beyond the comprehension of most college students, particularly those in two-year colleges. This approach leaves no middle ground between a "rigorous" treatment and an "intuitive" treatment. It is because of this lack of middle ground that the calculus has become "*a watered down, cookbook course in which all (the students) learn are recipes, without even being taught what it is that they are cooking*" (R. G. Douglas, 1986, p. iv). To avoid this dilemma, the trend has been to focus instead on *application* problems but this creates other difficulties. For example, it is quite natural for an economics instructor who has just defined the cost function to want to use the notion of derivative to define marginal cost. But it seems much less natural for a mathematics instructor to use the marginal cost to motivate the definition of the derivative. Presumably, if students are still at a point where they don't know what a derivative is, then neither can they be expected to know what a marginal cost is. Moreover, in our experience, instructors in client disciplines would rather we left their discipline to them and, in any case, this seems hardly the way to give the students a real appreciation of the calculus *per se*.

The calculus is currently in much the same situation as the internal combustion engine. It has been perfected over the past fifty years to the point where it performs its present function rather well but cannot accommodate the further demands of the "New Century". On the other hand, engines such as the Sterling, based on thermodynamic cycles quite different from that of the internal combustion engine, have not yet benefited from the current technology and thus have not yet been refined to the point where they can effectively compete with the internal combustion engine. However, such engines are being developed and *they* will meet the new demands. In like manner, there is a conception of the calculus, dating back to Lagrange (1797) which, though without, yet, the polish of the conventional approach, offers an alternative particularly well suited to those students who are not, at least a priori, favorably inclining towards mathematics.

In what follows, we present what we think is an alternative to the conventional three semesters Precalculus I - *Algebraic Functions*, Precalculus II - *Transcendental Functions*, Calculus I - *Differential Calculus*. It is an embodiment of Lagrange's approach for two four-hour-courses, **Differential Calculus I and II**. It was designed for students who, in the words of the NSF Request for Proposals for the development of a Calculus Curriculum, almost all "*choose not to continue their study of mathematics through calculus, thereby closing career options in*

Francesca Schremmer studied at the Université de Paris where she obtained a Diplôme d'Etudes Supérieures in Applied Mathematics. After being for several years a Researcher in Atmospheric Mechanics in the Department of Meteorology at the University of Paris, she came in 1965 to the US to study at the University of Pennsylvania where she wrote a doctoral thesis in 1972 on the Dirichlet problem for parabolic equations. She then worked on problems involving PDE arising from differential geometry. After teaching at Rutgers University and Bryn Mawr College, she has now been teaching since 1979 at West Chester University. Her interest remains in differential equations, ordinary and partial, particularly silutons and strange attractors.

Alain Schremmer studied at the Université de Paris where he obtained a Diplôme d'Etudes Supérieures in Applied Mathematics. He then worked for several years with the Piaget group, at the Institut Pédagogique National in Paris, while teaching at the Lycée Lakanal. In 1965, he came as a Fulbright Fellow to study logic, mathematical linguistics a model theory at the University of Pennsylvania where he is ABD. He has been teaching since 1966 at Community College of Philadelphia where he has developed several new courses and texts. He would like to return to Singular Perturbation Theory.

mathematics, engineering and the sciences." Many are women, minorities and "returning" adults[1].

Lagrange's Approach.

To illustrate the dilemma mentioned in the introduction, consider how we define continuity in the conventional approach

$$f \text{ is continuous at } x_0 \quad \text{iff} \quad \forall\varepsilon\exists\delta[0<|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon]$$

Even allowing for the use of ε's and δ's, one problem with *basing* the calculus on limits is that there is no procedure for *finding* them using the definition; one can only *check* whether or not a given candidate is the limit. In other words, there is only a *relation* between x_0 and the limit, and it is no more realistic to ask most students to come up with a candidate when the function is *not* continuous than with the requisite Skolem function $\delta=\delta(\varepsilon)$ when it is. Another problem is that it is difficult to prevent the students from identifing $\lim_{x\to x_0} f(x)$ with $f(x_0)$.

If, on the other hand, we decide, as in the usual "intuitive" presentation, to avoid ε's and δ's altogether, the problem then is that we are left without any definition. Continuity and differentiability basically become primitive terms whose understanding depends on whatever everyday connotations, if any, the words have for the students. Hence the gimmicks, electronic or otherwise, to buttress or create this "intuition". Most important, though, is that, even when *shown*, or even *experienced*, these concepts cannot be *used* and, eventually, must remain meaningless.

In the January-February 1988 issue of FOCUS, mention is made of a 1987 MAA Award for Expository Excellence, the George Polya Award for articles in THE COLLEGE MATHEMATICS JOURNAL, to I. C. Bivens, for "What a Tangent Line is When it isn't a Limit" (March, 1986 issue, pp.133-143). The committee's citation is quoted in part as: "*By defining the tangent line as the best linear approximation to the graph of a function near a point, [Bivens] has narrowed the gap, always treacherous to students, between an intuitive idea and a rigorous definition.* The subject of this article is fundamental to the first two years of college mathematics and should simplify things for students...." (Emphasis added).

Lagrange's approach is just the extension of this idea to best approximations of degree n. In other words, what we are really dealing with are jets of differentiable functions; we systematically approximate the value $f(x_0+h)$ of a function *near* the given point x_0 by a polynomial function whose degree depends on the problem at hand. Thus, Taylor polynomials are the basis of Lagrange's approach rather than being introduced at the end of the course, almost as an afterthought. We obtain them easily by truncating binomial expansions for polynomial functions and by division of polynomials for rational functions. In "all" other cases, we use the method of undetermined coefficients from the functional equation, algebraic or differential, of which they are the solution. Incidentally, even in the conventional approach, using the differential equation is by far the easier way for introducing transcendental functions. See, for instance, S. Lang (1976), Sections 4-1,2, and 3 or R. L. Finney and D. R. Ostbey (1984), Sections 4-8 and exercise 3.

We can read off all the usual notions of the differential calculus – continuity, differentiability, concavity, etc. – from the coefficients in the Taylor polynomial approximations. For instance, we have

$$f \text{ is continuous at } x_0 \quad \text{iff} \quad f(x_0+h)=f(x_0)+\ldots$$

and

$$f \text{ is differentiable at } x_0 \quad \text{iff} \quad f(x_0+h)=f(x_0)+lh+\ldots \text{ for some } l$$

and, if f is differentiable on an open set, we define $f'(x)$ as the function whose value at x_0 is l.

For example, given $f(x)=x^3$, we have $f(x_0+h)=x_0^3+3x_0^2h+3x_0h^2+\ldots$. Then, *by definition*, the value of $f'(x)$ at x_0 is the coefficient of h and that of $f''(x)$ is twice the coefficients of h^2. Note that the equation of the tangent at x_0 is $t_{x_0}(x)=x_0^3+3x_0^2(x-x_0)$ and that of the osculating parabola is $q_{x_0}(x)=x_0^3+3x_0^2(x-x_0)+3x_0(x-x_0)^2$. Also, just by looking at the graph of $f(x_0+h)=x_0^3+3x_0^2h$, we can *find* the *sided* limits at x_0. In fact, we can find sided limits this way even in cases which, in the conventional approach, require the use of L'Hôpital's rule.

Thus, while these notions appear to the students as disconnected topics on the conventional approach, they appear here as systematic features intrinsic to the successive Taylor approximations of $f(x_0+h)$. Moreover, these definitions can be used to *prove* the usual theorems quite simply by substituting the approximations for the functions. For instance, to prove the quotient rule, just divide $f(x_0+h)=A_0+A_1h+\ldots$ by $g(x_0+h)=B_0+B_1h+\ldots$. The coefficient of the first degree term in the quotient is the value of the derivative of $f(x)/g(x)$ at x_0.

The stage in the epistemological development of a person where she/he can *speculate*, that is operate *under assumptions*, is, in Piaget terminology, the abstract operational, or hypothetico-deductive stage. It is the stage required by and large by the conventional approach. For instance, a person at the hypothetico-deduc-

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mathematics, engineering and the sciences." Many are women, minorities and "returning" adults[1].

Lagrange's Approach.

To illustrate the dilemma mentioned in the introduction, consider how we define continuity in the conventional approach

$$f \text{ is continuous at } x_0 \quad \text{iff} \quad \forall\varepsilon\exists\delta[0<|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon]$$

Even allowing for the use of ε's and δ's, one problem with *basing* the calculus on limits is that there is no procedure for *finding* them using the definition; one can only *check* whether or not a given candidate is the limit. In other words, there is only a *relation* between x_0 and the limit, and it is no more realistic to ask most students to come up with a candidate when the function is *not* continuous than with the requisite Skolem function $\delta=\delta(\varepsilon)$ when it is. Another problem is that it is difficult to prevent the students from identifing $\lim_{x\to x_0} f(x)$ with $f(x_0)$.

If, on the other hand, we decide, as in the usual "intuitive" presentation, to avoid ε's and δ's altogether, the problem then is that we are left without any definition. Continuity and differentiability basically become primitive terms whose understanding depends on whatever everyday connotations, if any, the words have for the students. Hence the gimmicks, electronic or otherwise, to buttress or create this "intuition". Most important, though, is that, even when *shown*, or even *experienced*, these concepts cannot be *used* and, eventually, must remain meaningless.

In the January-February 1988 issue of FOCUS, mention is made of a 1987 MAA Award for Expository Excellence, the George Polya Award for articles in THE COLLEGE MATHEMATICS JOURNAL, to I. C. Bivens, for "What a Tangent Line is When it isn't a Limit" (March, 1986 issue, pp.133-143). The committee's citation is quoted in part as: "*By defining the tangent line as the best linear approximation to the graph of a function near a point, [Bivens] has narrowed the gap, always treacherous to students, between an intuitive idea and a rigorous definition.* The subject of this article is fundamental to the first two years of college mathematics and should simplify things for students...." (Emphasis added).

Lagrange's approach is just the extension of this idea to best approximations of degree n. In other words, what we are really dealing with are jets of differentiable functions; we systematically approximate the value $f(x_0+h)$ of a function *near* the given point x_0 by a polynomial function whose degree depends on the problem at hand. Thus, Taylor polynomials are the basis of Lagrange's approach rather than being introduced at the end of the course, almost as an afterthought. We obtain them easily by truncating binomial expansions for polynomial functions and by division of polynomials for rational functions. In "all" other cases, we use the method of undetermined coefficients from the functional equation, algebraic or differential, of which they are the solution. Incidentally, even in the conventional approach, using the differential equation is by far the easier way for introducing transcendental functions. See, for instance, S. Lang (1976), Sections 4-1,2, and 3 or R. L. Finney and D. R. Ostbey (1984), Sections 4-8 and exercise 3.

We can read off all the usual notions of the differential calculus – continuity, differentiability, concavity, etc. – from the coefficients in the Taylor polynomial approximations. For instance, we have

$$f \text{ is continuous at } x_0 \quad \text{iff} \quad f(x_0+h)=f(x_0)+\ldots$$

and

$$f \text{ is differentiable at } x_0 \quad \text{iff} \quad f(x_0+h)=f(x_0)+lh+\ldots \text{ for some } l$$

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tive stage would be willing to speculate on the consequences of *assuming* that $|x-x_0|$ is less than δ while persons at the concrete operational stage would absolutely refuse to do so on the ground that they don't *know* for a fact that $|x-x_0|$ is less than δ. By contrast, the concrete operational stage is the stage where a person can only *calculate*, that is operate *on existing data*. It is unfortunately the stage at which most students currently in college are (W. Palow, 1986). But, because the algebra prerequisites to Lagrange's approach are only those necessary to get the Taylor polynomials, it *"reduces(s) the demands on students for traditional manipulations of equations"*, (NSF, 1987), and thus allows us to serve these students too.

Content architecture

Given the problems encountered by our students, it seems most important that the degree of difficulty, as perceived by them, not increase exponentially as the course progresses. This is precisely what happens with the conventional precalculus - calculus architecture as it introduces a succession of concepts – limits and continuity, differentiability and derivate, etc. – *illustrated* with functions from a *"dictionary of functions"* presumably learned in an elementary fashion in some precalculus course (J. Mason & A. Schremmer, 1989).

By contrast, the deep significance of Lagrange's approach for the first year calculus is that it ties the conceptual development of the calculus to a hierarchy of functions reflected by the Taylor polynomial approximations: locally continuous functions are viewed as approximately constant, differentiable functions as approximately affine, twice-differentiable functions as approximately quadratic, etc. We are thus led to look at the differential calculus as the study of successive classes of increasingly complicated functions. In Differential Calculus I, we investigate **Affine Functions** ($f(x) = ax + b$), **Quadratic Functions** ($f(x) = ax^2 + bx + c$), **Homographic Functions** ($f(x) = [ax + b]/[cx + d]$). We start Differential Calculus II with **Power Functions** ($f(x) = \pm x^{\pm n}$) and **Binomial Functions** ($f(x) = \pm x^{\pm m} \pm x^{\pm n}$) and we continue with **Polynomial Functions, Laurent-Polynomial Functions, Rational Functions, Irrational Functions,** and **Transcendental Functions**. We treat each class completely before we move on to the next one. To achieve both a very lean calculus and a very sharp focus, all is eliminated that is not of *immediate* use in the sequel. Moreover, analytic geometry is almost entirely avoided and what is kept is only what can be recast in functional terms. For instance, rather than talking about the point-slope formula for a straight line, we solve the initial value differential problem $f''(x) = 0$ with $f'(x_0) = y'_0$ and $f(x_0) = y_0$. This Lagrangian architecture has powerful pedagogical advantages.

The first one is that the *level* of difficulty perceived by the students does not increase appreciably as they go on. What changes is the *nature* of the difficulties they encounter. During the first semester, the technical difficulties are very small and it is with the concepts themselves that the students have to cope. But the number of concepts needed initially is fairly small and the concepts themselves elementary. Moreover, the various features of a function near a point – sign, variance, concavity, etc. – are visualized and embodied by power functions seen as prototypes and then used as building blocks for "all" other functions. For instance, $p(x) = kx^{2n} (n > 0)$ is *really* the prototype of a local extreme as the approximation of a function as a local extreme will be of the form $f(x_0 + h) = A_0 + A_{2n}h^{2n} + ..., (n > 0)$. Affine functions have (almost always) one and only one zero, have no pole, and essentially only constant approximations. Their instant rate of change is easy to define since the average rate of change is the same between any two points. They have only a first derivative, are monotonic, have no extreme, no curvature, and the differential equations of which they are solutions are almost trivial. Then, quadratic functions still have no pole but can have two kinds of zeros or none at all. They have both constant and affine approximations. The instant rate of change changes linearly, and they have first and second derivatives. They have a turning point and therefore an extreme, and they have curvature but are "curling" everywhere; that is, have no inflection. The differential equations of which they are solutions are still almost trivial. Finally, homographic functions have a pole and two inflection points; one at the pole and one at infinity. They have constant, affine and quadratic approximations, etc. In general, these functions are particularly easy to discuss because they are just translated power functions, *globally* and *exactly*, while "all" other functions are only *locally* and *approximately* so.

As the students familiarize themselves with the concepts, they become progressively able to shift their attention to the technicalities involved in dealing with the more complicated functions of the second semester and to the new concepts that these require. This makes for a reasonably low learning gradient. At the very least, we are in a position where we can, ethically, demand that the students fully document their work (Schremmer, in press). In the best cases, it also introduces the students to the very basic mathematical temptation of wondering about generalizations such as: Affine functions have at most one zero; is this true of all functions? Affine functions change sign at zeros; is this true of all functions? Affine and quadratic functions do not change sign other than at zeros; is this true of all functions? The corresponding statement keeps having to be rephrased as the class of functions under investigation is enlarged. This is in marked contrast with the conventional approach where all statements are made from the outset in the most general terms possible, that is way beyond the students' experience and wildest expectations.

Conclusions

We started developing Lagrange's approach in 1982 with stand-alone alternatives to the conventional semesters of precalculus and differential calculus (F. Mattei & A. Schremmer, 1983, 1988 a). These were used on a regular basis, if more or less informally, by several

of our colleagues at Community College of Philadelphia. Then, in the Summer of 1988, Community College of Philadelphia approved the creation of a new Integrated Precalculus I - II & Differential Calculus two-semester sequence based on F. Schremmer & A. Schremmer (in press a, and 1988). Shortly afterwards, Community College of Philadelphia received a Calculus Development Grant to run a pilot and Essex Community College in Baltimore agreed to field test the sequence under the grant. While we have solid grounds to believe that our approach does work, at this point, we have made no attempt at statistical research.

The algebra requirements are minimal. Essentially, the students need only to be sufficiently proficient with polynomial algebra so that they can obtain the approximations. Anything beyond that is reviewed with algebra review cards (e.g., first and second degree equations and simple inequalities). As a result, we have no placement problems. Based on a five year statistical study of the standalones, we believe that the students who come out of our Basic Algebra course with an A, that is about 30%, will be capable of taking Differential Calculus I. To obtain an even more compact sequence, one should design a one semester Integrated Arithmetic and Algebra course. It should bring out the parallel between decimal/rational arithmetic and polynomial/rational algebra and it should promote a facility with decimal arithmetic and polynomial algebra to help test and appreciate orders of magnitude.

Since the integral calculus is almost entirely disjoint from the differential calculus, transfer of the two semester sequence does not pose any problem. In fact, as soon as we were awarded the grant, we contacted the three schools to which most of our students transfer and all three approved the sequence as equivalent to their Calculus I. Over the years, students who took the Lagrange Differential Calculus stand-alone have found themselves in our conventional second semester Integral Calculus together with students from the conventional Differential Calculus. The only way we seem to be able to tell them apart is from their reaction to L'Hôpital's rule; students coming from the Lagrange based Differential Calculus fail to see what the problem is with, say, $\lim_{x\to 0}[1-\cos x]/x^2$ since their reaction is, in any case, immediately to approximately $\cos x$ by $1-x^2/2+\ \ldots$!

The only problem is in transferring Differential Calculus I alone. As this is written, the end of the second semester in which we have been offering Differential Calculus I, conversations and negotiations continue to be carried out with neighboring institutions to convince them to try similar courses themselves. Several have already joined us in a new NSF grant proposal. This would obviously solve the problem of Community College of Philadelphia students transferring before they can take Differential Calculus II. But, in any event, students with Differential Calculus I should certainly do quite well in any Precalculus II. In fact, Differential Calculus I has already been accepted by one institution as equivalent to their precalculus. A more delicate problem is to obtain credit for "short calculus", defined as polynomial calculus with a heavy reliance on cookbook manipulations.

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AUTHORS NOTES. The systematic use of polynomial approximations is of course well known, in Perturbation Theory, as the method of asymptotic expansions. But other than as such, we do not know of any reference. We invite anyone interested in Lagrange's approach, in our project, and/or in getting copies of our materials to contact us.

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The Introductory Statistics Course of the 1990s

Mary R. Parker

Austin Community College, Austin, Texas 78714

Easy access to computers has changed the practice of statistics dramatically in the 1980s. Not only is it easier to collect and store data, but it is much easier to carry out the computations necessary for elaborate statistical analyses. To some extent, people with less formal statistical training are performing these sophisticated statistical analyses, and many more of us are being asked to understand, or at least accept, the results of such analyses as relevant to our decision making. In this environment, statistical literacy is a most important skill.

The challenge to us, as we create the introductory statistics courses of the 1990s, is to focus on the aspects of statistical literacy that will be most useful for interpreting statistical analyses. This includes determination of whether the data are adequate to answer the questions of interest, whether the techniques are appropriate for the data and the questions of interest, and whether the results of the application of the techniques are interpreted correctly. In this context, in many cases it is important to teach students how to perform the computations involved, because an understanding of how to use a technique is often of great help in determining when the technique should be used. However, we must be careful not to overload the course with discussion and exercises on computations, at the expense of discussion and exercises on selection of techniques and interpretation of results.

Computers in Statistics Courses

Most of the statistical analyses a person will meet in the world of the 1990s and beyond will be of sufficient complexity for a computer to be necessary, or at least convenient. Our students should encounter some of these analyses on real data in their introductory courses. In fact, with the widespread availability of microcomputers and a variety of statistical software, students should be taught to perform some simulated experiments and carry out some analyses on a computer. Whether to use a powerful package such as MINITAB or SPSS, or one of the less powerful, more tutorial packages is a question that must be decided by each school in much the same manner as textbook selection. Whatever package is used, it is crucial that the computer assignments focus on the primary goals: understanding the concepts of statistics, selecting appropriate techniques for an analysis, and interpreting the results.

Components of Statistical Literacy

The great majority of students in an introductory statistics course will never take another statistics course. Therefore, we must teach them to use some techniques and provide some practice in analysis in order to contribute to their statistical literacy. Three other components of statistical literacy are crucial and also deserve attention. We need to teach the material in such a way that students recognize the limitations of their knowledge; that they know how to find readable, useful information to help them extend their knowledge; and that they recognize questions and problems for which it might be useful to consult a professional statistician.

Before computer software made it relatively easy for anyone to perform complicated statistical computations, teachers of statistics could assume that students would have been exposed to a description of how to determine whether a given statistical technique is appropriate before they would be able to use it. Clearly that is no longer true. Sending out our introductory statistics students with some knowledge but no feeling for the limitations of that knowledge and no guidance on how to extend their knowledge will cause even more problems for them and for society than it would have ten years ago.

Many statistics teachers have found that assignment of student projects, sometimes including outside reading, to be quite helpful. Sellers (1989) provides some excellent suggestions about integrating projects into the course, and Anderson and Loynes (1987) provides detailed explanations of 36 sample projects, many suitable for an introductory course. Tanur, et al. (1978) is a collection of essays illustrating past accomplishments and current uses of statistics and probability. Many of these essays do not assume any prior statistical knowledge. Questions suitable for assignment are given after each article.

Mary R. Parker received her MA from the University of Oregon in 1972, and her PhD in mathematics, with a concentration in statistics, from the University of Texas at Austin in 1988. She has taught mathematics and statistics at Austin Community College for eleven years.

of when it is to be used. Furthermore, introduction of nonparametric techniques for use when the assumptions for the parametric techniques are not met reinforces the emphasis on checking assumptions. Iman and Conover (1983) has much more discussion of nonparametric techniques than most introductory statistics texts. Because nonparametric tests are in the same chapter as the similar parametric techniques, it is easy to emphasize how they answer similar questions but do not require the same assumptions.

Throughout the discussion of inferential techniques, the emphasis should be on what assumptions are necessary and how to verify these assumptions. Computer generated graphics make this easier than ever before. For example, our discussion of the constant variance requirement for predictions using the regression model will be much more meaningful when students understand how to use a residual plot to check it. In cases where we assume the population is normal or essentially normal, we can routinely do a normal plot of the data. Discussion of the difference of robustness of the various techniques against violations of this assumption of normality will have more reality to the students when they are confident of being able to assess how closely the data fits the assumption.

Particular Topics

How much probability should an introductory statistics course include? While we all would like our students to know more of everything, students in a one-semester statistics course are best served by a short and informal presentation of the ideas of probability. The ideas should include conditional probability and very little problem solving should be expected. Basically, they need to understand a probability statement about a sampling distribution. I have never felt that extensive homework on "urn" problems or their equivalent contributes much to that understanding.

The discussions of sampling in most introductory statistics books focus on how to obtain a representative sample. A problem with ending the discussion with that topic is that we are missing one of the main opportunities to point out the limitations on the knowledge available from an introductory statistics course. The other major issue in sampling is to use a sampling scheme that gives an estimator with as small a variance as possible. Then the sample can be as small as possible (as cheap as possible). We will certainly not tell students how to compute variances for any sampling scheme except simple random sampling; but if they know that it can be done, they will be able to find someone to tell them how to do it when it matters.

A good introductory statistics course certainly includes regression. However, two important topics that are frequently not mentioned are how statisticians handle nonlinear relationships and multiple explanatory variables. Of course, a thorough understanding of these topics would require several additional courses, but one or two additional class meetings devoted to basic ideas of how linear regression is adjusted to incorporate these considerations and how to interpret the results could be some of the most valuable time in the course for many students. When one is asked to understand a regression analysis out in the world, it is seldom a simple linear regression.

For some time, many introductory statistics courses have emphasized hypothesis testing to a greater degree than have more advanced courses, or than have practicing statisticians. At the same time, it is not clear that students gain a thorough enough understanding of the concept of the power of tests to be confident about the true meaning of the tests. I would like to say that, in the 1990s, we will teach our students techniques for choosing a test and sample size with high power against an appropriate value in the alternative hypothesis. But I don't see movement in that direction. Rather, I expect that we will see a decreasing emphasis on hypothesis testing, and an increasing reliance on estimation. Certainly the need to include new material in the course will require decreased emphasis on some topics, and hypothesis testing is likely to be one of those.

Conclusion

The introductory statistics course of the 1990s will be a more conceptual and broader course than that of the 1980s. Its main purpose will be to develop the ability within students to choose, apply, and interpret appropriate techniques for analysis of data. Descriptive statistics, especially computer-generated graphical techniques, will be a larger part of the course than before. Because of a recognition that students will be able to apply, through computers, many more techniques than they have mastered in an introductory course, there will be a strong emphasis on an overview of statistics, and on increasing the ability to learn about new techniques through self-study.

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A Fractal Outline of a Fractal Course

Avery Solomon

Cornell University, Ithaca, NY 14853

Mandelbrot coined the term "fractal" from the latin *fractus* meaning to create irregular fragments (Mandelbrot, 1982). The term describes a family of shapes which are fragmented, yet have a tendency toward a property called "self-similarity" or "scaling"—meaning "the degree of their irregularity is identical at all scales."

Although the special terminology of fractals is new, the roots of fractal theory lie deep in a variety of basic ideas available to most students with only a modest high school mathematics background. However, most of the concepts developed in understanding the idea of a fractal are solid mathematics, and tie in a tremendous range of topics. A central focus of the related activities is getting students to explore the relation of the finite and infinite, using both discrete and continuous iterative and recursive operations.

In addition to interrelating many strands of basic mathematics, fractal theory touches on some of the most current research on physical phenomena in biology (such as how cellular organisms grow and how they are nourished and breathe), dynamical systems (such as damped harmonic motion and weather) and structures (such as stress in building materials and structure of coastlines). Since many of the results have been possible only with the use of computers, it is not surprising that so many of the fractal concepts are innately suited to computer investigation, both numerically and graphically. Sources for software are given in the Appendix.

The purpose of this article is to present an outline of a fractal course with pieces blown up in greater detail to reveal some of the internal complexities. Sections I and II explore in some detail the roots of fractal theory in geometry and arithmetic. The mathematical concepts are highlighted in bold print as they unfold through the topics. Some *activities* for students are also presented. Section III outlines further topic areas which build on the prior foundations and Section IV suggests alternative approaches for a course.

Avery is currently director of the Cornell/Schools Mathematics Resource Program, and teaches in the Cornell Math/Science Teacher Education Program. Many of the ideas for approaches to topics in this article were developed by the author and Professor David Henderson of the Cornell Department of Mathematics, for a series of inservice mathematics courses for high school teachers. This work was funded by a Title II grant from New York State. There are also plans being made by the author for a summer fractal institute in 1990.

I. The View from Geometry.

Fractal geometry looks at shapes which are reached in stages by a limit process. The first unit of a fractal course could examine the features of this geometry in four stages:

*A. **Self-similarity.*** Constructing stages of a fractal set involves the idea of **self-similarity** and **recursion.** A simple first curve is the Koch Snowflake.

Begin with an equilateral triangle. At each new stage, add an equilateral triangle to the middle third of each previous segment.

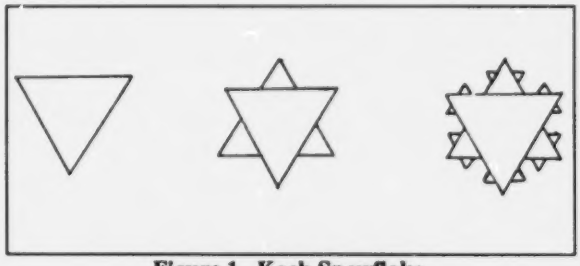


Figure 1. Koch Snowflake.

The snowflake's perimeter is multiplied by 4/3 at each stage. Thus, the perimeter approaches infinity. However, the area stays bounded. Each stage adds on four times as many triangles as there were in the previous stage. The area of each of the new triangles is always 1/9 of the area of a triangle from a previous stage. Therefore, the area enclosed by a Koch Snowflake is 8/5 of the area of the original triangle. (See Esgard, 1988 for a detailed discussion.)

When the recursive processes are continued indefinitely, questions come up which can lead students to explore **limit processes,** discover or review **geometric sequences** and **series,** and raise issues about the relationship of different **dimensional measures.**

Activities: Construct several stages of other self-similar figures. Write programs or use commercially available software to draw several stages of the process. Look up the construction of "collage sets" in Chapter 6 of Barnsley (1988). Draw some shapes in which pieces similar to the

original are taken away rather than added. Look up some three-dimensional analogies (Barcellos, 1984).

B. The Relationship of Surface Area and Volume. Most interesting physical examples occur at the next dimensional level, in which the ratio of **surface area** and **volume** are important. As animals grow or evolve larger while keeping the same shape, the ratio of their surface area to volume gets smaller – thus making the exchange of nutrients, oxygen and sunlight more difficult. To put it another way, for a given volume we may change the ratio of surface area to volume by changing the shape. What are the minimal and maximal surface areas, and what creature or situation demands either one?

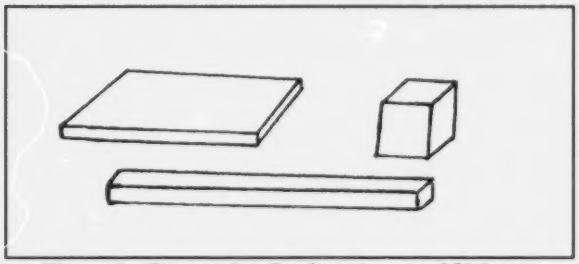


Figure 2. Comparing Surface Area and Volume.

In general, a minimal surface area is needed by things which don't want to interact with their environment. A maximal ratio, however, is needed by things which do want to interact. If the organism or process can spread out indefinitely, there is no finite upper bound – for a fixed volume there is no limit to the surface area possible. But if the spread of an organism is constrained, it naturally approaches a fractal boundary.

Activities: Explore the relationship of surface area and volume and maximum and minimum values using cubes of varying sizes. Compare SA and V as the edges get larger. Make graphs/tables, and especially graph the ratio of SA/V vs. edge length (see Fleishman, 1987). Compare SA/V for several rectilinear shapes with the same volume (see Figure 2).

Put a drop of water into a small tube of oil. This models a noninteractive situation. Put a drop of ink in a glass of water. Here the "organism" wants to interact.

Find out how coastlines are measured. Investigate how tree roots, our lung system, or the capillary systems which must deliver blood to every cell are laid out to maximize exchange. Read the Stevens (1974) chapter on growth from a point. Read the article on "Being the right size" in Haldane (1985).

C. Scaling and Dimension. Geometrically, "fractal" refers both to a kind of set, as well as a process in which infinite complexity is brought into focus by recursion and self-similarity. If we imagine the iterative process taken to its limit, we have a set which is said to be **self-similar**; that is, magnifying or **scaling** a part of a "true" fractal we get a set which is identical to the original. For example, if we magnify a side of the Koch Snowflake by 3, we get 4 of the original sets.

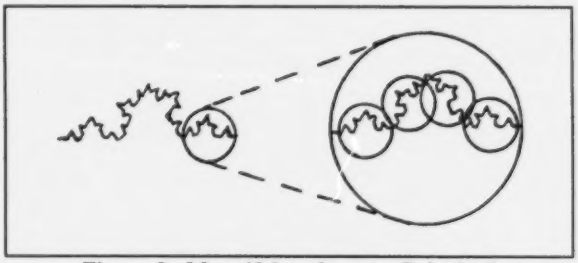


Figure 3. Magnifying the snowflake by 3 and you get 4 of the original.

We can describe the measure of a **fractal boundary** (what Mandelbrot, 1982, calls the "degree of irregularity") by using this self-similarity property, through a use of **dimensional analysis**. A picture analysis of integer dimensions leads to the notion of a **self-similarity dimension**, and **fractional dimensions** for sets.

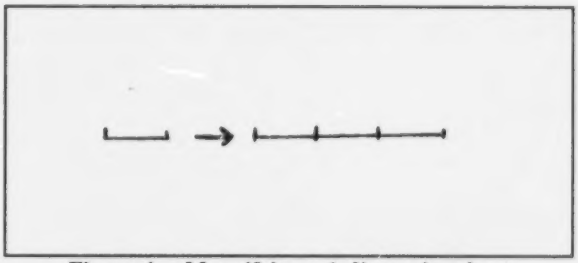


Figure 4a. Magnifying a 1-dimensional set by 3 increases the measure (length) by 3. $3^1 = 3$.

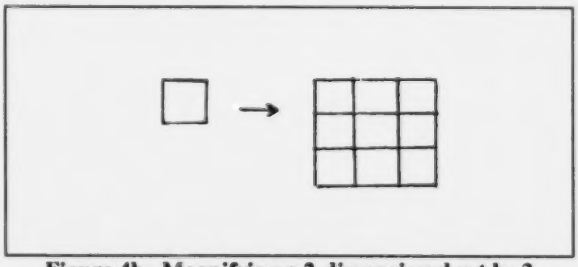


Figure 4b. Magnifying a 2-dimensional set by 3 increases the measure (area) by 9. $3^2 = 9$.

In general, if D is the dimension, 3 the magnification, and X the increased measure, then 3^D should equal X. Using this principle on the snowflake, we get $3^D = 4$, and $D = \frac{\log 4}{\log 3} = 1.26$. Another example of magnification 3 is given in Figure 5.

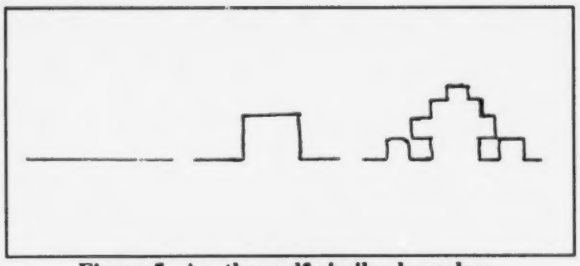


Figure 5. Another self-similar boundary. $3^D=5$. $D=1.46$.

Mandelbrot uses this kind of scale factor to describe the complexity of coastlines of different countries. He also gives a technical definition of a fractal as a set whose self-similarity dimension is not equal to its **Hausdorff dimension** (Mandelbrot, 1982; also see Barcellos, 1984). Exploring the measures of these sets leads to a discussion of **measure theory**.

The Cantor middle third set is of special interest because it is a fractal set which is discrete rather than connected. The set remaining after removing the middle thirds has dimension log 2/log 3 = 0.63.

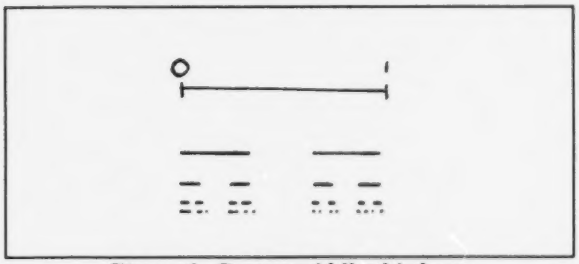


Figure 6. Cantor middle-third set.

There are many other intricate examples of fractal sets, such as the Sierpinski Carpet, and space filling curves like the "flowsnake" – a linear type object which has self-similarity dimension 2 (see Mandelbrot, 1982 and Barnsley, 1988).

Activities: Try to cover the Cantor set, or the rational numbers on [0,1] with a cut up piece of string. What is the minimal length required (see Barnsley, 1988)?

D. Curves and Smoothness. The sheer complexity of a fractal boundary raises questions involving **continuity, smoothness** and **infinitesimals**.

Activities: Is a fractal boundary a broken line? Is it composed of an infinite number of infinitesimal segments? Is it continuous? You cannot draw the boundary of the Koch Snowflake – it is infinitely far between any two points on it. It is not smooth – at any point.

The idea of **magnification** provides a thought model useful not only for looking at the complexity of a fractal, but also for exploring concepts which become essential in **differential calculus** as well.

Activities: Look at different features of graphs through a "microscope":

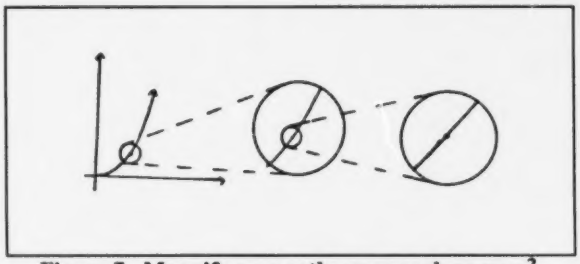


Figure 7. Magnify a smooth curve such as $y=x^2$. It becomes a straight line.

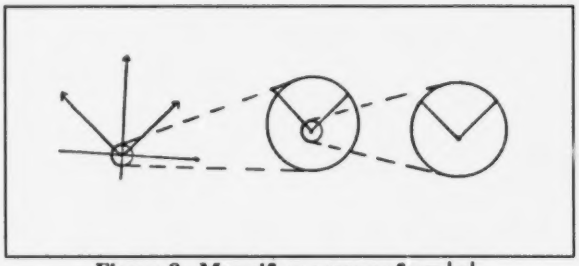


Figure 8. Magnify a corner of $y=|x|$. It never straightens.

Figure 9. Magnify a fractal. It remains locally self-similar.

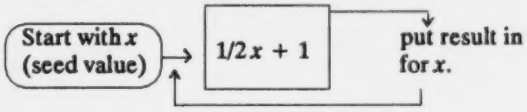
In an article by the author (Solomon, 1988) the application of the idea of magnification to the real numbers and the nature of lines is explored in detail.

Activities: Is there a smallest particle? Is a line a set of infinitesimals?

Imagine that within each cell of our body are atoms, and within the nucleus of each atom are worlds with beings whose cells have atoms, with worlds within and so on. Imagine that our whole universe is within a nucleus of a cell of a creature who is within a universe, and so on. What new presuppositions do you need to add to Euclidean geometry in order to think about infinite shapes?

II. The View from Arithmetic

We next look at the arithmetic analog of the geometric process of iterating a simple procedure. Consider iterating the rule: "start with a number, take half of it, and add 1."



There are three interrelated approaches to this process: through substitution, transformations, and graphing.

A. Iteration by Substitution. This topic involves **composition, series and convergence.** Students can study linear iterations by simply substituting values for a few stages.

Activities:

1. Using a calculator, start the procedure $x \rightarrow .5x+1$ with seed number 1 and repeat 5 times. Make a table. Describe the pattern.
2. We say that a rule "converges" if the sequence of numbers generated "converge"; i.e., the numbers in the sequence approach or get closer to a limiting value. Does this procedure converge?
3. Write a short program and start the procedure with other numbers: 1/3, 0, −5, 20. Let it run for 5, 10, 50 iterations. What is happening and why? Does the procedure converge for these initial values?
4. Change the rule to some of the following, and run with different seed numbers for each rule. What happens?

$$.2x-3 : -.4x+3 : 2x-1 : -5x : x-3 : -x+1$$

5. Try iterating quadratic functions.

Investigation confirms the experimental results that for the general rule $Ax + B$, the iteration involves a geometric series, and converge to limit $B/(1-A)$ if $|A| < 1$ independently of the initial value of x; diverges if $|A| > 1$; and $A = 1$ is of special interest.

Iteration is a powerful numerical tool to solve problems and equations. Some further applications are taken up in Section IV.

B. Transformations, Real Number Operations and Fixed Points. A linear function $f(x) = Ax+B$ can be interpreted as a **transformation** of a line. Adding is equivalent to translating, multiplying is scaling, negating is a half-turn of the line. Figure 10 presents the results of stages of $x \rightarrow -.5x+3$.

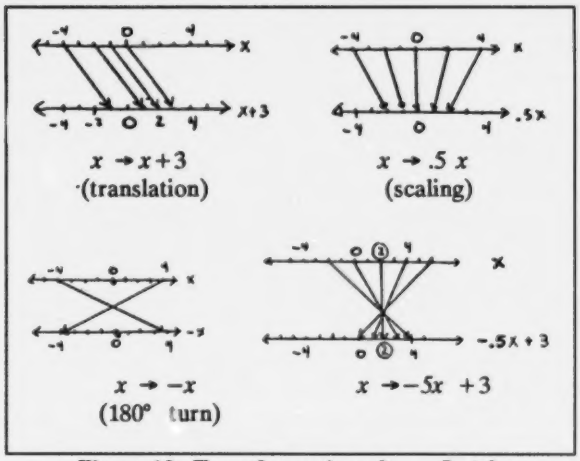


Figure 10. Transformations for $-.5x+3$.

If $0 < |A| < 1$, the transformation has a **fixed point**; a value P for which $P = AP + B$. P is seen to be equal to $B/(1-A)$, which is the limit of an infinite geometric series. If $0 < |A| < 1$, the linear rule can thus be described as shrinking the whole line by A towards center $P = A/(1-B)$ at each iteration. (If A is negative, the rule also rotates 180 degrees as it shrinks). If $|A| = 1$, we do not get a magnification, only a translation and/or rotation. $|A| > 1$ has divergent effects.

Activities: Find the fixed points of linear rules. Investigate the fixed points of nonlinear rules. Investigate the symmetries of a line and other plane figures. Write a program to find the square root of a number by the guess, divide and average method. Try very large or small first guesses. Try negative numbers. What happens?

Note that the interpretation of **real number operations** as transformations carries over to **plane (complex) numbers**, and is worth a whole unit in itself. For example, you might explore the idea of plane numbers as vectors, operations defined as transformations, and sequences and series of complex numbers (see Henderson & Solomon, 1985-88).

C. Graphical Interpretation, Composition and Attractive Points. Linear iterations can also be approached graphically. Students can explore the procedure which makes use of the line $y=x$ to iterate the rule $Ax+B$ starting from any point (a,0) on the x-axis. For example, start on the graph of $f(x)=\frac{1}{3}x+1$ at the point x_0 equal to (−3/2,0). Move vertically to (3/2, 1/2) and then horizontally to (1/2, 1/2) to get the new x-value 1/2. Apply f again by moving up to (1/2, 7/6). Continue.

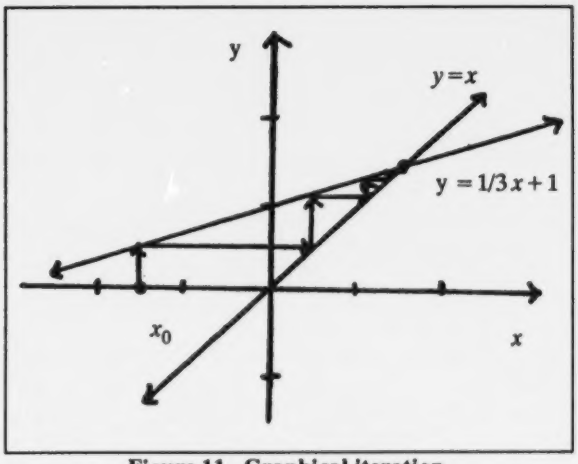


Figure 11. Graphical iteration.

This procedure is a graphical composition of f with itself, f ∘ f ∘ f ∘ ... and generalizes to any compositions.

Students can investigate different values of A. It is not hard to see that this process converges iff the slope A is such that $|A| < 1$, and the point of convergence P is the intersection point of $y = f(x)$ with $y = x$ at $B/(1-A)$. Point P is called an **attractive fixed point** if all x values nearby are attracted (converge) to it by the iterative procedure. In the linear case all x-values converge to P if $|A| < 1$. If $|A| > 1$ then any seed except $B/(1-A)$ will diverge away from P, and P is called a **repulsive fixed point.**

Activity: Iterate other functions using a graphical procedure.

Moving from the linear to nonlinear functions provides many interesting examples. Allowing complex starting points will tie the numerical iterations back to the fractal geometry.

III. More Advanced Fractal Topics

A. Iteration of Nonlinear Functions. The quadratic case, $x^2 + C$, gives very interesting results. For certain values of C, iteration of $x^2 + C$ converges. For other C, some seed values cycle, others converge to different attractors, or diverge.

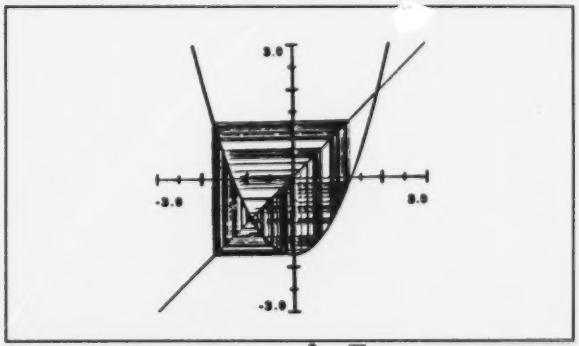


Figure 12. Iterating $f(x) = x^2 - \sqrt{3}$ from $x_0 = 1.240$ on ANALYZER program.

Graphical composition also leads to **iterative methods for solving equations** such as $2^x = x^2$.

Activities: To see what happens when iterating $x^2 + C$ in a range of values, write a program to do the following: Plan an axis of C values in $[-2, 0.5]$ in .05 increments, vs. an axis of x values in $[-2,2]$ with increments of .08. For each C plot the values to which each seed value of x in $[-2,2]$ converge after 50 iterations (see Hubbard and West, 1989).

B. Dynamical Systems and Chaos. Many natural phenomena can be modelled as repeated applications of simple equations governing the change of state of the phenomena from one moment to the next. There is often an extreme **sensitivity to initial conditions**; that is, a minute error in the beginning leads to a catastrophic error at the end. This has been termed the "Butterfly Effect," from the possible effect just one butterfly flapping its wings might have on the weather in an area.

Sensitivity may be seen in the following example of using Newton's method for finding roots (see Figure 13).

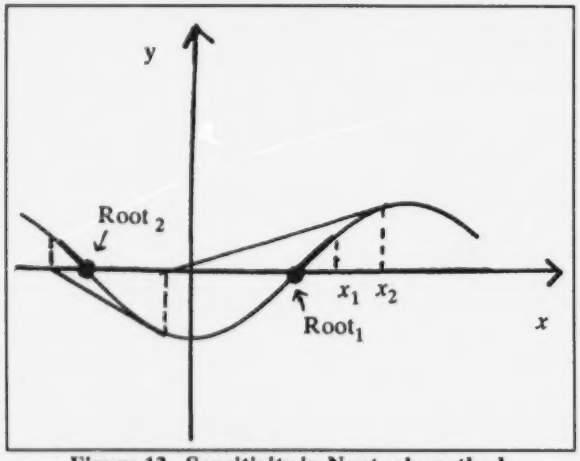


Figure 13. Sensitivity in Newton's method: close seed values find different roots.

From a few basic laws iterated indefinitely, nature seems to build up random phenomena. Gleick (1987) is an excellent guide through some of these ideas. The accompanying mathematics can be filled in according to the level of the students. Recent applications of these ideas to **chaotic behavior in dynamical systems** include population models, weather and medicine. An application of **graphing attractors** is modeling stable wildlife populations from a graph of year-to-year growth. A 1989 Nova special on chaos showed several applications. The whole issue of *Cornell Engineering Quarterly* listed under Hubbard (1986) is worth careful reading. Mathematical connections can be made to **vector fields** where the effect of starting at different points leads to different paths (see

Hubbard & West, 1989), and such far- reaching topics as **Markhov chains** (see Sloyer, 1987).

C. Complex/Plane Number Transformations. The geometrical definition of plane operations in terms of transformations forms a unique approach to **complex numbers, Demoivre's theorem, polar coordinates, vectors,** and many **trigonometric formulas.** It also leads to a discussion of **geometric series of complex (plane) numbers** interpreted as series of **similar figures,** and **fixed points of planar transformations** (see Henderson & Solomon, 1985-88 and Figure 14).

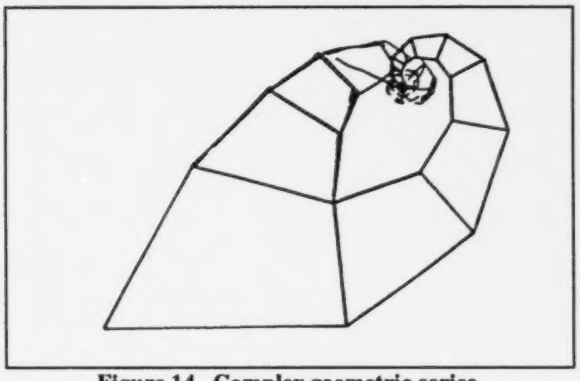


Figure 14. Complex geometric series.

D. Mandelbrot Set. Iterating **complex-coefficient** functions takes us into the heart of fractals. Even in the iteration of quadratic complex functions, the set of seed-values which lead to convergence is a fractal, and each distinct equation must be considered separately.

For each complex number C, the iteration of $Z \rightarrow Z^2 + C$ converges for some seed values of Z, and not for others. The set of convergent values is called the **Julia set** for C, and has a fractal boundary. In general, we can put all the complex numbers C into two sets: those C for which the Julia set (converging seeds) is connected, and those C values for which the Julia set is not connected. An example of a connected Julia set is given in Figure 15.

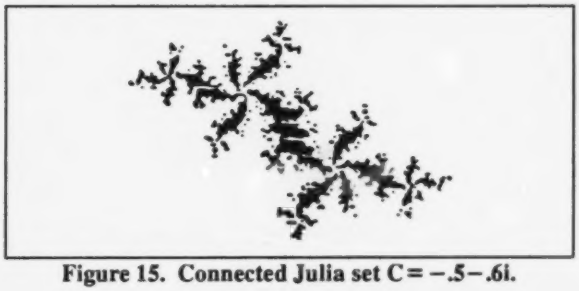


Figure 15. Connected Julia set C = −.5−.6i.

The fabulously complicated **Mandelbrot set** can be characterized as the set of all complex numbers C whose Julia sets are connected. it is also characterizeable as the set of all C for which 0 is in the Julia set (converges).

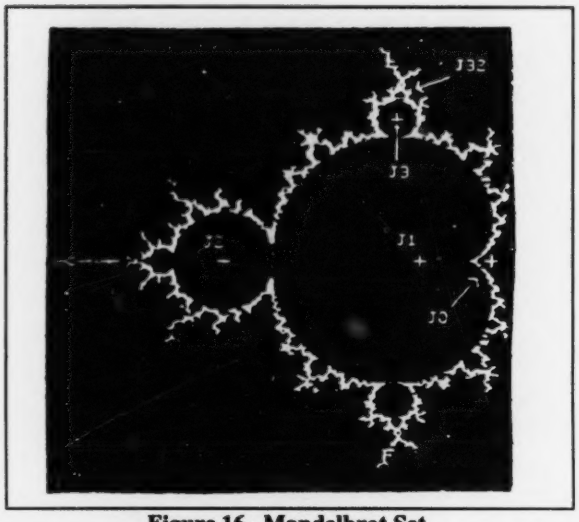


Figure 16. Mandelbrot Set.
[from advertizing for *Art Matrix*]

Zooming in on the Mandelbrot set (see Figure 16) at different levels reveals different complex configurations, so it is not strictly self-similar. It is itself connected—having tiny filaments running off, with small Buddha-like figures embedded in them (see Peitgen & Richter, 1988 and 1989, and Hubbard, 1986).

IV. Fractal Courses

A fractal course could have several orientations: toward the liberal arts, toward mathematics or science majors prior to calculus, or towards those with a one-semester or year calculus background.

A course for liberal arts students could easily use Gleick's (1987) excellent book as an outline, taking a deeper look at the mathematics according to the ability of the students. Supplemental material from Hofstader (1979), Ricker (1982), Brunton (1988), and Thomas (1984) provide worthwhile philosophical side readings.

A very formal course could be based on Barnsley (1988), or on articles and parts of books as outlined above and in the references.

Many of the topics and ideas would fit easily into existing courses in computer application, finite mathematics, precalculus, and calculus, if one's program did not justify a separate course. In addition, a selection of topics could give a different focus to a computer programming course. Peitgen and Saupe (1989) is an excellent resource for such a course.

Historical connections would make an interesting part of a fractal course. Noteworthy are the ideas of Escher in art (see Coxeter, 1986), D'Arcy Thompson in morphology (see Stevens, 1974), and in the study of the

universe (see Morrison, Morrison, & Eames, 1982). The Greeks used the idea in the golden ratio (see Huntley, 1970) and the question about whether one can divide matter or a line infinitely traces back even earlier. A recent conversation with a Tibetan artist-monk confirmed that Tibetan mandalas also make use of self-similarity properties.

Follow-up activities in more advanced courses include problems involving simple **differential equations, complex series, vector fields, dynamical systems** (see Hubbard and West, 1989).

Summary

The topic of fractals provides an opportunity for students to investigate and tie together a wealth of related ideas. Especially important is that the topic provide fresh insights for understanding limits, continuity and smoothness, among other calculus ideas which explore the boundary of the finite and the infinite. Above all, these topics provide a rich range of new approaches to mathematics and many opportunities for students to take an active part in learning.

Appendix

Sources of Slides, Videos and Pictures

Aerial Press. P.O. Box 1360, Santa Cruz, CA 95061. Ph. 408-425-8619. (Chaos and the designer of fractal software.)

Art Matrix. P.O. Box 880, Ithaca, NY 14851. (The original fractal videos and slides. Ask about Mandelzoom software.)

Cornell University. Department of Mathematics, White Hall, Ithaca, NY 14853. (Source of "Analyzer," an inexpensive MacIntosh program to explore graphs, including iterating and composing functions.)

Sintar Software. P.O. Box 3746, Bellevue, WA 98009. (Inexpensive Apple and IBM software for the Mandelbrot Set.)

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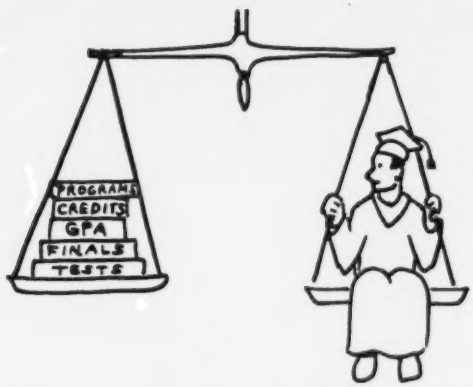
Section 3.
Special Programs



Every Student Deserves A Fair Chance

- Strategies for Teaching Remedial Mathematics to Learning Disabled Students – by Juliana Corn and Alexandra Klein

  "Far from being a homogeneous group, learning disabled students are a very diverse population. Each student brings a unique pattern of strengths and weaknesses to the learning situation and effective teaching must address these individual differences."

- A One-Semester Arithmetic Remediation Corequisite for College Level Mathematics – by Pansy W. Brunson

  "It is imperative that colleges provide a high quality remediation program that responds to the needs of their students."

  "What was needed was a program that would remediate but not require students to take an additional mathematics course."

- Increasing Communication in College Mathematics Classes – by George Spanos and Deborah Short

  "...it is obvious that lack of proficiency in the language of instruction has deleterious effects upon the ability of students to deal with mathematics texts, word problems and lectures."

Strategies for Teaching Remedial Mathematics to Learning Disabled Students

Juliana Corn and Alexandra Klein

Queensborough Community College, CUNY, Bayside, NY 11364

Queensborough Community College, an "open admission" two-year college that is part of the City University of New York (CUNY), has seen a rise in the number of students who come to college lacking skills in arithmetic and elementary algebra. All entering students take a placement examination and approximately 80-85% fail it. To serve these students, a five hour per week, 14 week remedial mathematics course is required before any credit-bearing mathematics courses may be taken.

Recently, it has become apparent that a particular subgroup of students who fail the placement examination has been experiencing virtually no success because of learning disabilities. To provide equal access and reasonable accommodations for these students, which is mandated under Section 504 of the Rehabilitation Act of 1973, the Mathematics Department has registered only learning disabled (LD) students in two specified sections of the remedial mathematics course since 1986. Learning disabled students were identified by the Office of Disabled Student Services with the aid of documentation supplied by the students. The diagnosis of learning disabilities is based on a comprehensive psycho-educational battery of tests and is not to be confused with retardation, lack of motivation or the effects of poor teaching. Under the auspices of a federal grant (Corn, 1985) for 1986-1989, a multi-faceted approach to teaching remedial mathematics was developed and put into place. The teaching and tutoring strategies discussed in this article are an outgrowth of "A Demonstration Project to Teach Remedial Mathematics to Students With Learning Disabilities" and reflect the philosophy and practices of its staff.

Dr. Juliana Corn, professor and chair of the Mathematics Department at Queensborough Community College from 1981-1989, served as director of the project described. She has taught mathematics for thirty years and is the coauthor of a technical mathematics text.

Ms. Alexandra Klein, MA, was the learning disabilities specialist for the project. She has had more than eighteen years experience in evaluation and remediation of learning problems, as well as staff supervision and training.

Project Structure

The staff of the project consisted of two mathematics faculty, four to six peer tutors, a project director, a learning disabilities specialist, and part-time administrative/clerical staff. For the first two years, three other mathematics teachers worked on evaluating, acquiring and creating CAI (computer-assisted instruction) packages to supplement classroom instruction.

The teachers, tutors and CAI staff were trained by the learning disabilities specialist prior to each school year. They also met with the project director and learning disabilities specialist throughout the school year to discuss specific student needs and problems, and to plan instructional strategies.

Students received help from the tutors one hour a week in the classroom and one hour a week outside the class in small groups. They were also referred to the college's Instructional Resource Center to use the CAI materials, as well as interactive videotapes created by the project director to coordinate with, and reinforce each classroom lesson. They practiced specific topics with which they were having difficulty or reviewed for exams.

The project's formal objectives were:

(1) To modify existing community college remedial mathematics curriculum (MA-010) to meet the specific educational needs of the students with learning disabilities.

(2) To develop a complete package of computer assisted instruction (CAI) materials to enhance and supplement each topic in remedial mathematics curriculum.

(3) To provide comprehensive support services to students with learning disabilities who are in the process of completing the remedial mathematics requirements.

(4) To provide technical assistance, consultation and specialized training to a core faculty in order to develop instructional strategies appropriate for teaching mathematics to students with learning disabilities.

(5) To develop a handbook of remedial mathematics instructional methods and strategies in order to aid faculty in the instruction of students with learning disabilities.

(6) To develop a package of video tapes in the basic mathematics curriculum to be used to enhance and supplement each topic in the remedial mathematics curriculum.

(7) To disseminate information to college mathematics faculty and other professionals, both secondary and postsecondary, via workshops and presentations at professional conferences.

To understand the teaching and tutoring styles which were emphasized by the project's staff, it is necessary to describe the characteristic difficulties of learning disabled students and their effects on learning mathematics.

Characteristics of Learning Disabled Students

Far from being a homogeneous group, learning disabled students are a very diverse population. In fact, mathematics is an area of strength for some LD students. Each student brings a unique pattern of strengths and weaknesses to the learning situation and effective teaching must address these individual differences.

Like other Queensborough students who take the remedial mathematics course, those with learning disabilities may have difficulty understanding mathematical concepts or remembering algorithms. However, mathematics also requires the coordination of many additional skills which, in the general population, we assume to have been automaticized long before adulthood.

Weaknesses in these underlying skills, including the ability to form and align numbers, rote knowledge of multiplication facts, spatial and temporal orientation, language processing (oral or written), and the ability to focus and sustain attention, may result in a multiplicity of "careless" errors, even in the presence of conceptual understanding. For example, a student who must "figure out" the product of 9 and 8 as part of a signed number problem may not notice upon returning to the problem that his task was to multiply 9 by *negative* 8. If this student also has difficulty with spatial planning, he might construct the 9 times table in the middle of the problem rather than on scrap paper, thus obscuring his work and interfering with his ability to proceed.

Attention deficits, including distractibility, impulsiveness, perseveration and poor self-monitoring skills, are often especially frustrating to students and teachers alike. Students who tune in and out perform inconsistently; they may be able to do a problem one day but not another. Unfortunately, these students often suffer when their teachers conclude that they would always do well if only they would try harder (Levine, 1987).

Tutoring Strategies

Individualized tutoring is an important component of any program for LD students because it affords the opportunity to examine the process by which a student has arrived at a particular answer. Individualization can be accomplished in a small tutoring group if the tutor rotates among the students and assists one at a time while the others practice new skills independently.

Student participation is a key element of the tutoring process. Active student involvement facilitates memory and attention and allows the tutor to observe the students' working patterns. One way to encourage active participation is to ask direct questions which help the students to focus their attention and to reflect on what they are doing. Questions such as "What kind of problem is this?" or "Why do you have to find the LCD?" are preferable to vague inquiries such as "Do you understand?" Good questions will also help the tutor to detect areas of difficulty.

While the error patterns of remedial students can often be determined by examining their completed algorithms, it is sometimes necessary to actually *observe* a student with learning disabilities in order to determine the source of a particular error. Consider the following example in which the student was asked to write $6\frac{13}{12}$ as a mixed number with a proper fraction. He wrote:

$$6\frac{13}{12} + 12 = 1\frac{1}{13} = 7\frac{1}{3}$$

Direct observation revealed that the student correctly divided 13 by 12, verbalized "12 and one left over is 13" but "accidentally" wrote the 13 rather than the 12. Finally, the answer $7\frac{1}{3}$ resulted from incorrect copying of the denominator 13. Further investigation revealed no conceptual difficulties, and the student was able to correct his errors independently. In this case, teaching self-monitoring strategies would be indicated while clarifying the properties of improper fractions and mixed numbers would be superfluous.

Teaching Strategies

In contrast to the one-to-one tutoring situation or even the small group instruction mode, classroom teaching demands more global strategies to satisfy the needs of the students. Even a section limited to students with learning disabilities, requires a combination of approaches so that each student may benefit. Understanding the problems of the LD students will enable teachers to vary their strategies to reach viable solutions. *Flexibility* of teaching style should be the trademark, with nothing being too silly or obvious to try, as long as it hits the mark.

Because learning disabled students often have problems with time management and organizational details, the course structure and requirements should be clarified. Explicit instructional objectives and a detailed syllabus are a must. Grading procedures, classroom behaviors and expectations should be explained. Office hours, test and tutoring schedules (if available) should be given in written form.

Another helpful approach, which is often neglected by college teachers, is explaining the structure and effective use of the textbook. Mathematics books have certain formats which students need to understand in order to make efficient use of them. For example, the facts that sample problems appear before the homework exercises; that summaries of formulas, algorithms, rules and definitions are generally at the end of a section; and that answers to some of the homework problems appear at the end of the book should all be discussed with the class. A homework problem should be chosen and the appropriate behavior in using the textbook to solve the problem should be modeled by the teacher.

Since learning disabled students may be inconsistent in their day-to-day knowledge and skills, and often have memory problems (Alley & Deschler, 1979), it is vital to begin each class with a summary of the previous lesson. As Gagne (1970) has described it, one of the conditions for concept learning is the learner's previous knowledge, on which the new concept is to be based. An overview of the topic sometimes helps to anchor the specific discussion which follows. In a similar fashion, a quick summary of the lesson should be given at the end of the class. All points should be made in both verbal and written form.

Language and reading skills are usually major sensitive spots for students with learning disabilities (Cherkes-Julkowski, 1985). Accordingly, all vocabulary, symbols and signs of operation should be explained in a simple, consistent manner. Lengthy, complex sentences should be avoided, as well as overly technical or sophisticated language. Always use the same phrases to explain the same situations. What is only a synonym or literary effort to the instructor may represent ignorance and chaos to the student. A frequent source of frustration for the student who is attempting the homework is confusion over textbook instruction (e.g., "simplify" may not be well-defined in his mind). It is worthwhile to take a moment of class time to clarify homework language.

Consistent notation, explanation and procedure should be employed in each learning situation. For example, in solving linear equations of the form $ax + b = cx + d$, one opening gambit should be taught (e.g., subtract cx from both sides of the equation), rather than leaving it to the student to choose among the four initial approaches. This helps the student to deal with his impulsiveness, disorganization and tendency to try random trial and error strategies (Alley & Deschler, 1979). It also helps in focusing on the correct part of the problem.

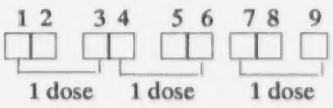
To aid the student in note-taking, all steps for an algorithm or procedure should be written on the board, together with a sample problem illustrating it. Any shortcuts, mnemonic devices or work arrangements which will help the student in memorization, recall or monitoring errors should be shown explicitly. A special way of utilizing this material might be to have the student use individual index cards for each task attack, organization and follow through. Anyone who has seen the notebooks of some of these students will understand the advantage of compartmentalizing each procedure.

Since some students with learning disabilities cannot retrieve information efficiently, mere rote memory will not be the best approach. Concepts, applications and the underlying development of the procedures should be taught (Ausubel, 1968). For example, one may teach division of mixed numbers (e.g., $4\frac{1}{2} + 1\frac{1}{2}$) merely by telling the student to change each mixed number to an improper fraction and proceed as in division of fractions. To a student with learning problems, it makes just as much sense to remember $4 + 1 = 4$ and $\frac{1}{2} + \frac{1}{2} = 1$ so the answer may be $4 \cdot 1 = 4$ or perhaps $4 + 1 = 5$.

On the other hand, by a discussion of the following problem, the meaning of the division procedure may be explained.

Problem: You must take $4\frac{1}{2}$ tablets of medicine each day. Each dose must be $1\frac{1}{2}$ tablets. How many times a day do you take the medicine?

Solution: A diagram like this encourages the idea that $4\frac{1}{2} = \frac{9}{2}$ and $1\frac{1}{2} = \frac{3}{2}$.



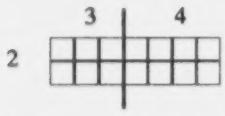
Thus, the intuitive notion that the answer, 3, results from partitioning the nine halves into subsets of three halves, helps pave the way for the formal procedure $\frac{9}{2} \div \frac{3}{2} = 3$

This problem also illustrates the idea that concrete descriptions and materials should be used to explore symbolic structures (Fitzmaurice Hayes, 1985). For instance, one might use real or play money to develop addition of signed numbers. Positive numbers represent gaining money, while negative represent owing money. The students should be allowed to perform a number of transactions, involving examples of all the rules, before developing the formal rules. This will "cure them" of

such catch phrases as, "two negatives make a positive" and other longstanding errors.

Another effective use of "real-life" knowledge which helps students understand and remember a formal arithmetic and algebraic algorithm is adding coins of the same and different denominations. For example, one dime plus two dimes equals three dimes or $\frac{1}{10} + \frac{2}{10} = \frac{3}{10}$. One dime plus one quarter *doesn't* equal two "dime-quarters." What does it equal? The immediate response from students is that the answer is thirty-five cents and this enables the teacher to motivate the concepts and procedures for adding fractions with unlike denominators. The remark that the answer is also seven nickels leads, naturally and reasonably, into a consideration of *lowest* common denominators. From there it is but a hop and skip to algebraic problems.

The distributive law of multiplication over addition or subtraction may be made "real" by considering the construction of tile patio floors, as follows:

> The overall measurements of a patio are two feet by seven feet.
>
> 
>
>
> How many square tiles of one foot on a side are needed for the floor? If you have a divider as shown, how many tiles are needed for each part?

This enables the student to see that $2(3+4) = 2 \cdot 3 + 2 \cdot 4$. More important the diagram may be generalized by replacing the "3" by an "x" or the "4" by a "y", or each number by a letter. Expressions of the form, $a(b\text{-}c)$ and $x(3+4)$ also may be clarified by this type of explanation. The *latter* leads easily into the consideration of adding like terms.

Other strategies which are helpful for students with perceptual problems include: using arrows to describe the flow of work; underlining each group of like terms with a different symbol; writing parts of problems such as exponents with colored chalk or pencil to emphasize them; separating subprocedures, such as finding the LCC, from the main flow of the problem by boxing them; and using lined paper turned sideways to align digits or like terms. Further strategies aimed at focusing the student's attention and blocking irrelevant stimuli have been developed through an analysis of the underlying difficulties.

An essential aspect to the teaching-learning situation is the active involvement of the student (Greenwood & Anderson, 1983). "Hands-on" practice in copious amounts serves to reinforce each concept and provides continuous feedback to the student and the teacher. A small amount of lecture goes a long way. Two-way communication between class and teacher is preferable. Students should be encouraged to go to the board, as well as to ask and answer questions from their seats. A non-judgmental, relaxed response to the questions, as opposed to a "you should know that already" remark is advised.

Spiral assignments (i.e., giving homework on old topics, as well as the new work each day) are particularly helpful to these students. Frequent short quizzes and overlapping cumulative exams reinforce the effects of spiral homework.

Conclusion

The teaching and tutoring strategies discussed here have been developed under the auspices of a federally-funded project to aid LD students in learning arithmetic and algebra. A teacher's guide, a software review, handbook, CAI software and videotapes have been produced and are available at minimal cost to any educational agency. For further information, write to Prof. Sandra Peskin, LD Math Project, c/o Math Dept., Queensborough Community College, Bayside, NY 11364.

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A One-Semester Arithmetic Remediation Corequisite for College Level Mathematics

Pansy Waycaster Brunson

Western Kentucky University, Bowling Green, Kentucky 42101

Since 1978, the percentage of college students enrolled in remedial or "developmental" courses has increased substantially. In a national survey, Akst (1986) found that 86% of the responding colleges offer remedial mathematics programs. In a more recent report, Lynn Steen (1988), Chair of the Conference Board of Mathematical Sciences, said that "up to three-quarters of all students enrolled in mathematics courses in colleges and universities are already being taught skills they should have learned in high school." It is imperative that colleges provide a high quality remediation program that responds to the needs of their students.

At Western Kentucky University students may satisfy their general education mathematics requirement in different ways. Some major programs require College Algebra, while General Mathematics (a survey of topics – set theory, logic, consumer math, geometry, probability, and statistics) suffices for other areas. If a student lacks arithmetic skills and is on the College Algebra track, remediation is accomplished through Introductory Algebra (a noncredit course that includes a review of arithmetic concepts). For a student enrolled in General Mathematics (Math 109) prior to the Fall, 1986 semester, there was no departmental procedure for arithmetic remediation; competency in arithmetic was considered a prerequisite for General Mathematics. However, many students who signed up for Math 109 were deficient in arithmetic. What was needed was a program that would remediate but not require students to take an additional mathematics course. The purpose of this article is to describe our solution – the AST Program.

Pansy is Director of the Mathematics and Science Center and supervises the Arithmetic Skills Test Program at Western Kentucky University. She received her PhD in Adult Education from Indiana University, Bloomington, an MA in Mathematics Education from Eastern Kentucky University and BA in Mathematics from Berea College. Her current interests include the areas of developmental mathematics and returning adult students.

The AST Program

The Mathematics Department Developmental Committee developed an arithmetic remediation program, and implemented it during the fall of 1986. The basic objective of the Arithmetic Skills Test (AST) Program is to remediate those Math 109 students who are weak in arithmetic skills without adding another semester course of mathematics to their degree program. To achieve this objective, any necessary arithmetic remediation must be executed during the semester in which the student is enrolled in the course. Consequently, the AST Program was designed to be completed during the first eight weeks of any semester. As competency in arithmetic is a prerequisite for Math 109, students must pass the AST within this eight-week period to remain enrolled in the course. Thus, this eight-week *corequisite* program is an alternative to spending a whole semester in an arithmetic course before taking Math 109. The scenario of this eight-week program follows:

AST(a): At the first class meeting of Math 109, each student receives a sample test as a guide for preparing for the AST. One week later all Math 109 students take AST(a). Tests are graded and diagnosed for each student's area(s) of weakness.

Skills Sessions: Students who do not pass AST(a) are given a diagnosis sheet that includes their AST(a) score as well as the areas in which they need remediation. The sheet also includes a schedule of when help sessions are to be held. Should their work or class schedules preclude their attending these help sessions, modules for all four of the content areas – fractions, decimals, percents and measurement – may be obtained for individual work. Free tutorial assistance on these modules is available in the Mathematics Laboratory. This help is offered separate from and in addition to their regular Math 109 class.

AST(b): After four weeks of structured Skills Center Sessions, students who did not pass the AST(a) are per-

mitted to take AST(b). Tests are again graded and diagnosed, and then returned to the students.

AST(c): The Skills Center remains open for unstructured (individual) sessions each afternoon for one more week. AST(c) is given during the eighth week of classes. Any student who has not passed the AST by the end of the withdrawal period may withdraw from the course with a grade of W (*or else receive a grade of F in the course*). Figure 1 summarizes this eight-week program.

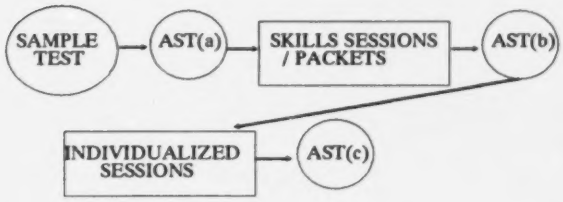

Figure 1. The AST Program.

Results

Test results for the Fall, 1986 through Spring, 1989 semester show that an average of 62% of the students passed AST(a). Approximately half of the students who attempted AST(b) or AST(c) successfully passed that test. Over this three-year period, an average of 84% of the Math 109 students passed the AST on one of the three offerings, marking the success of the program. Students who had not passed the AST and did withdraw from Math 109 (16%) could enroll in a biterm arithmetic course to remain full-time students. They would then spend the remainder of the semester working toward mastery of arithmetic concepts and retake Math 109 next term.

One concern about the AST program is its impact on the success rate in Math 109. Remember, the primary goal of the AST Program was to remediate students weak in arithmetic skills—*not* to improve their success rate in Math 109. Although it was hoped, even expected, that better arithmetic skills would bring about a greater success rate in Math 109, this is a question which *cannot* be answered in this study. That is, since most Math 109 students who had not passed the AST by its last offering withdrew from Math 109 rather than receive an F in the course, it is impossible to compare their grades with the grades of those students who did pass the AST. However,it is possible to look at a related question—"Has the AST Program had an effect on the grade distribution for Math 109 for those students completing the course?" To address this question data on students' grades were obtained for five semesters before and five semesters after the AST was initiated and a chi-square test for homogeneity was performed. The data appears in Table 1. With a p-value of approximately 0.11, the difference between the grade distributions before and after the AST is not statistically significant.

	A B C	D	F
Before AST	1633 71%	340 15%	323 14%
After AST	1590 71%	376 17%	284 13%

Table 1. Math 109 grades.

Although the grade distribution has not changed appreciably, the withdrawal rate has increased dramatically. The average withdrawal rate before the AST was 12.8% and the average withdrawal rate after the AST is 24%. A look at the AST grades for students withdrawing from Math 109 reveals that about half withdrew because they did not pass the AST; the other half withdrew even though they *had* passed the AST. What does this statistic imply? Speculation offers two possibilities: (1) students enrolling in Math 109 for the past five semesters are weaker in arithmetic skills than students prior to this period; and/or (2) students in semesters prior to the AST Program were passing Math 109 without competency in arithmetic skills. Aside from speculation, a higher withdrawal rate should not be surprising if one considers the increased workload on these students who are already weak in mathematics. However, the continuation of the AST program is justified to assist those students who are able to put forth this extra effort to avoid an additional semester of remediation in mathematics.

Recommendations

Based on the findings from this study, the following recommendations for the Arithmetic Skills Test Program are:

1. Meet the skills session four days per week for two weeks rather than twice a week for four weeks. The purpose is to prevent procrastination on the part of the students in getting needed arithmetic help. Skills sessions records for Fall, 1986 and Spring, 1987 show that attendance for the latter two concept areas (percents and measurement) was higher than for the first two concept areas. Having the skills sessions concentrated in a two-week period immediately before the next AST should encourage the developmental students to attend *all* sessions and not postpone needed attention to arithmetic.

2. Offer a biterm course and a full semester course in arithmetic for those students who are unable to pass the AST. This insures that students who are unable to pass the AST and do withdraw from Math 109 have a recourse.

3. Do a follow-up study of the progress of those students who enrolled in an arithmetic course and then retook

Math 109 to determine the effectiveness of these arithmetic courses.

Summary

In summary, the AST program has achieved its basic goal of helping to remediate arithmetic weaknesses in Math 109 students without adding another semester course to their degree program. In addition, the AST program has highlighted the need for a regular course in arithmetic for nonscience majors to prepare them for taking the General Mathematics course. Also, a biterm course has been designed for Math 109 students who attended the AST skills sessions, yet were unable to absorb the necessary content in the allotted time frame. Regardless of the avenue for arithmetic remediation—AST program, biterm course, or full semester course—most important is the fact that these various routes are available and that they provide students who are weak in arithmetic skills some flexibility to incorporate the needed remediation into their programs.

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AUTHOR'S NOTE: Any reader interested in receiving sample forms and/or further information about the AST Program should contact the author.

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Increasing Communication in College Mathematics Classes

George Spanos and Deborah Short

Center for Applied Linguistics, 1118 22nd Street, N.W., Washington, D.C. 20037

The Problem

Imagine you have received the opportunity to study mathematics pedagogy at a teacher training center in France. On the first day of class your teacher presents this exercise to increase your sensitivity to learning mathematics in a second language. "The following word problem requires no complicated mathematical formulas," says your instructor. "Work with a partner for about two minutes and solve it." Anxious to succeed, you turn to the classmate seated next to you and begin to study the problem:

> Jean et André sont des frères. Jean est l'aîné. Les deux assistent à des cours au lycée qui est à moins de cinq kilomètres de leur maison à Paris. Bien que la différence d'âge entre eux soit de trois ans, celle de leur niveau scolaire n'en est que deux. Jean est en quatrième. En quelle classe André, est-il?

Neither you nor your partner is fluent in French, but your knowledge of English cognates such as *différence* enables you to translate enough words to determine what operation is needed to find the solution. Some of the terms are unfamiliar and some of the phrases are confusing, but after the two minutes elapse, you are ready to tell the teacher, "André is in second grade."

The instructor does not announce the solution right away, but instead distributes a worksheet with an English version of the problem. The instructor explains that there are several factors involved in problem solving. Some relate to mathematical knowledge, some to language skills, and others to knowledge about the world.

The students are given the following English translation with three questions:

> Jean and André are brothers. Jean is older. The two attend a high school which is less than five kilometers from their home in Paris. Although the difference in their ages is three years, the difference in their grades at school is only two. Jean is in the fourth grade. What grade is André in?

1) What are the language difficulties in this problem?
2) What are some mathematics difficulties in this problem?
3) What are some extra-linguistic features that could cause difficulty in solving this problem?

Students are given five minutes to study the translation and answer the questions. One student offers the opinion that the whole problem is difficult because the language is French. "Since I don't speak French, each word is hard." Many students concur, and thus, begin to realize the innate difficulties word problems pose for non-native speaking students. The instructor then lists some possible language difficulties:

> difficult lexical items, such as *aîné, niveau*;
> false cognates, such as *assistent;*
> conjunctions, such as *bien que*;
> comparative terms or structures, such as *moins de*;
> grammar structures with relative and subordinate clauses, such as *qui est à moins de cinq kilomètres de leur maison*; and *bien que...*
> unfamiliar forms, such as *n'en est que, soit.*

Although the operation needed to find the solution is not difficult, the class discovers some potential mathematics difficulties in the wording of the problem. The instructor lists such things as extraneous information; i.e., too many unnecessary numbers, and a mixture of cardinal (*deux, trois*) and ordinal (*quatrième*) numbers.

At this point, the instructor gives the answer: 6th grade. You are surprised. When you had seen the English version, it reaffirmed your belief that Andre was in second grade. After all, 4−2 = 2. Several students raise their hands to challenge the instructor, who, in turn,

George is a research associate at the Center for Applied Linguistics, where he directs a project aimed at improving articulation between language arts classes and mathematics and science classes.

Deborah is a researcher, teacher trainer, and materials developer at the Center for Applied Linguistics. She is currently coordinating a project for improving articulation between language arts and mathematics and science classes.

focuses attention on the last question on the worksheet, which asks the students to list some of the extralinguistic features of the problem.

Finally, one student, who has some familiarity with the French education system, says, "I forgot they lived in France. The sentence with that information seemed extraneous, so I ignored it." Another interjects, "You're right. We have to be familiar with the French school system."

The instructor agrees. Indeed, simply knowing the language and the required mathematical skills may not be sufficient to solve problems. There can be cultural issues as well. In this problem students need to know that the French system counts its grade levels backwards (by American standards), from 6th to 1st in secondary school. One cannot assume that all students have the same background knowledge as someone who has studied in a particular educational system.

Focusing on the Language of Mathematics

The preceding example suggests the desirability of instruction which is sensitive to the linguistic and cultural needs of language minority students, or students for whom the language of instruction is not their native or primary language. From the language educator's point of view, it is obvious that lack of proficiency in the language of instruction has deleterious effects upon the ability of students to deal with mathematics texts, word problems, and lectures. Many language educators (e.g., Spanos, Rhodes, Dale & Crandall, 1988) and a growing number of mathematics educators (e.g., Pimm, 1988 and Cuevas, 1984) are providing arguments which suggest that the nature of mathematics language imposes a heavy load on all students regardless of the language of instruction.

In 1984, researchers from the Center for Applied Linguistics (CAL) (Crandall, Dale, Rhodes, & Spanos, 1984) initiated a project funded by the Fund for the Improvement of Postsecondary Education (FIPSE). This study involved collaborative research with mathematics educators at several two-year colleges with high language minority enrollments and led to the development of a set of materials which could be used as a language-focused supplement to beginning algebra classes. The research phase of the project involved group problem-solving activities with language majority and language minority students to examine the nature of mathematics language and student difficulties in dealing with this language. Not surprisingly, the researchers found considerable evidence that the performance of some language minority students was severely impeded by lack of familiarity with basic conversational English, academic English, or both. These findings corroborated the findings of mathematics educators who had been working with Hispanic college students (*e.g.*, Mestre, 1981). What was unexpected, however, was the fact that native speakers of English faced comparable difficulties with the language of mathematics. Also surprising, was the fact that, despite awareness on the part of many of the mathematics instructors who participated in the project, there were few, if any, language-based materials or activities in mathematics classrooms and fewer opportunities for language arts teachers to become involved in the task of educating these students. In short, there was little, if any, articulation between language arts programs and mathematics programs, despite the fact that large numbers of students enrolled in mathematics had obvious language deficiencies.

The following excerpts shed light on the language problem. They are selected from a transcript involving a group of three freshman beginning algebra students at a two-year college in a Rocky Mountain state. The three students, S1, S2, and S3, have just finished trying to solve the word problem below and have been asked by the researcher, R, to explain the reasons for any difficulties they might have had. S1 and S2 are native speakers of English, while S3 is a language minority student from Mexico. They were asked to talk aloud while trying to solve the problem and were later interviewed by R, who tried to elicit information about the connection between language and problem-solving ability.

Two cars which are 375 miles apart and whose speeds differ by 5 miles per hour are moving toward each other. They will meet in three hours. What is the speed of each car?

S1: Well, I really don't know why we did everything the way we did, because I was just trying to simplify it in my mind, and I found it difficult to figure out which – what I'm supposed to do with these figures. What is in relation to what. Part of what we did, what I started doing, was to draw it out, in a picture like that. And I realized that the two cars had to come together, you know. Then I realized what would it take, sort of off track, what would it take for one car to go that whole distance.

R: So you had to picture it?

S1: Yeah, I tried to visualize it.

R: Is that what you normally try to do with these types of problems?

S1: I normally try to avoid these kinds of problems.

S3: These types of problems are the hardest.

R: Picturing makes it hard though? Is that the problem?

S3: Yeah, cause if you read a problem like in mathematics symbols, it's all written down. All you have to do is fill in the formula. If you have to picture it and write it down yourself, that's harder.

R: Yeah?

S1: Like I said before, it's the interpretation, the wording, and what relation. Without it being written already numerically, or even if it's already with a letter symbol there you don't know a formula. You've got all these words mixing it up. You take all this crap

out and you've got 375 divided by 3 equals *x* times or...

R: Was there anything in particular in the wording in this problem, anything specific that you can point to that's very important?

S1: "Moving *toward* each other." OK, that throws a big clinch in it. How're you going to figure out exactly when they are going to meet? I started thinking *where* are they going to meet.

R: So, *where they're going to meet* is beside the point?

S1: Yeah, but still, you know how fast is one car going and the other car going to get to the point. These things are just designed to be confusing.

R: Is there anything that could make them simpler?

S3: If they did the three hours time, it would be – that was the key.

S1: Is there another way we could have done it? I don't know.

S2: There probably, I mean, it would come out the same if you got it right, but I don't think you could have varied it too much. The problem is really limited. You have to be very specific with what you are doing with your numbers. In any math problem, you can't – like in English you can play with different words, but in math you have to be very specific.

The students exhibit some anxiety regarding the design of word problems, yet they also demonstrate a remarkable awareness of the linguistic difficulties inherent in the problem. The native English speakers, S1 and S2, are quick to point out the obstacles for representing the problem mathematically, citing such things as the interpretation, the general complexity of wording, difficult phrases such as *moving toward each other*, and the fact that mathematics language is more specific than ordinary English. They are quite adept at discussing the problem (i.e, their basic English conversational skills are highly developed), but this was of little help to them in arriving at a solution equation, a task involving the application of, among other things, academic language and reasoning skills. (In fact, the students arrived at the solution equation only after guessing that the cars would be going between 50 and65 miles per hour and finding an equation to fit their guess.)

The non-native speaker, while cognizant of the linguistic difficulties, was relatively silent in the larger transcript from which these remarks were culled. This was due, in part, to the fact that his basic English language skills were limited in comparison to the fluent, often sarcastic, style of his partners. But part of the reason was also that he seemed to understand the problem and was frustrated in his attempts to get the other students to take note of the time factor. In his final comment above, S3 points out the need for information regarding the amount of time (three hours) in order to solve the problem. It became evident that this student had an understanding of the "distance equals rate times time" formula and was trying to get the other students on task toward an appropriate solution equation. The other students knew from their algebra class that they would need to incorporate variables in order to set up an algebraic equation, but they were oblivious to the importance of the given information regarding the time, opting instead to focus on the distance and the fact that there were two cars. Had S3 been successful in communicating the importance of the time factor, it is possible that the students would have found a quicker route to the solution.

This raises an important issue. While basic language skills are important, they are not sufficient for S1 and S2 to solve the problem. A good foundation in basic concepts and problem-solving strategies is also needed, coupled with the ability to communicate and utilize these concepts and strategies in academic settings. In contrast, S3 seemed to have the requisite mathematical knowledge, but his communication skills were not developed to the point where he could get the attention of his peers. In this sense, S3 is not very different from any student (or even a teacher!) who knows the subject matter but is unable to verbalize problems effectively. This point suggests that any talk of increasing communication in the mathematics classroom must be coupled with the realization that students (and teachers) need to be taught to apply their language skills in appropriate and effective ways.

Research by Cummins (1981) and our own work with college students supports this view. Language minority students are often quick to develop the social language skills which enable them to communicate with their peers outside of the classroom. Cummins refers to such skills as "basic interpersonal communication skills." Within an academic context, however, this basic proficiency is inadequate for several reasons. One reason involves inexperience or lack of understanding of terminology and writing styles particular to a content area. Another reason is lack of preparation to perform the higher order language and cognitive tasks required in rigorous academic content courses. This latter point is applicable to native speakers of English as well, who are often not skilled in analysis, argumentation, evaluation, etc. Such skills require what Cummins refers to as "cognitive academic language proficiency."

Linguists such as Halliday have suggested that mathematics has a specialized register which students must be trained to recognize and use successfully in their mathematics courses. A linguistic register is "a variety of language that is oriented to a particular context, to a certain type of activity, involving certain groups of people, with a certain rhetorical force" (Halliday, 1975). In a recent article on this subject, Spanos, Rhodes, Dale, and Crandall (1988) provide a chart which describes some of the semantic, syntactic and pragmatic features that make the mathematics register problematic for limited English proficient students. For example, the lexical item *binomial* might be a new vocabulary word a student must learn, while *square* might be a familiar word (the

geometric shape or even slang for conservative) with a new denotation (a number times itself). A syntactic features such as word order in a word problem can play a critical role in a student's solution process. Thus, the statement *there are three times as many eggs as chickens* might be represented incorrectly as $3e = c$, instead of the correct equation $3c = e$. Finally, pragmatic phenomena, such as lack of familiarity with sales tax rates, discounts, mark-ups, etc., might result in student inability to handle tax rate problems, which often presuppose such knowledge.

Meeting the Communication Need

Teacher Training and Materials Development. In recent years, training workshops and seminars have been conducted at selected sites nationwide. These workshops have provided mathematics teachers with the opportunity to consider language objectives and increased communication in their classes. For instance, CAL staff recently conducted a week-long institute on integrating language and content instruction for faculty from several New Jersey community colleges where the improvement of academic language skills has become a major concern. An important aspect of these training seminars, as well as those given by colleagues at other institutions, is the joint participation of mathematics educators and language educators. This allows for cooperative activities that draw on the expertise of both disciplines. The training seminars present teachers with the theoretical background for integrating language and content and provide opportunities for application through the analysis of word problems, suggested instructional strategies and techniques, and assessment tools. The techniques include discovery learning, hands-on and problem-solving activities, cooperative learning and group work, and peer tutoring, as well as additional methods to make mathematics more relevant to the students' lives.

Two training videos are currently under production (see Resources) which will demonstrate the mathematics/language approach and materials. Accompanying manuals can be used by teachers for self-instruction when direct training is unavailable. As a component of a multifaceted workshop package, a video is appropriate for college training programs which seek to address the articulation problem between secondary school and college classes.

Once teachers have been trained to increase communication in class, they will need appropriate materials to develop lessons and activities. In some instances, they can attend workshops on materials adaptations where they learn to modify existing materials to suit particular needs. In such workshops, strategy sheets (see Resources) which focus on content and language objectives in lesson plans designed with communicative activities are used as developmental models. In other instances, teachers might consider using prepared supplemental materials, such as *English Skills for Algebra* (Crandall, Dale, Rhodes, & Spanos, 1989), which help students gain proficiency in the academic language of mathematics through interactive listening, speaking, reading, and writing activities. A glossary (see Resources) which provides a structured listing, annotations, and suggested instructional strategies for key mathematics terminology is also available.

Assessment of Mathematics Language Proficiency. Although assessment tools for determining concept mastery of mathematics are numerous, instruments to measure mathematics language proficiency are scarce. Still, some measures are currently being developed and field tested such as the *Pre-Algebra Lexicon* (see Resources). The diagnostic techniques in the lexicon are organized according to four mathematics categories (concepts, operations, word problems, and problem solving) and the four language skills (listening, speaking, reading, and writing). These techniques allow teachers to assess the growth of language skills within the context of daily mathematics instruction. The following example, from the writing and operations categories, shows one technique for assessing writing skills with algebraic expressions:

> Directions: Look at the algebraic expression on the left. Use the correct words to complete the phrases on the right which are verbal "translations" of the symbolic expression. An example is done for you.
>
> Example: $x + 7$ a. the *sum* of x *and* seven
> b. x *plus* seven
>
> 1. x^2 a. x _____
> b. the _____ of x
> 2. x-12 a. x _____ _____ twelve
> b. twelve _____ _____ x
>
> 3. $x^2 + y^3$ a. x _____ plus y _____
> b. the ___ of x ___ the ____ of y

These samples are representative of those items teachers can construct from their own textbooks. When students have completed the task, teachers should list the terms used incorrectly and develop instructional activities to reinforce the correct usage. Such activities could include a paired exercise where one student, acting as tutor, quizzes another student on the correct ways to read/write symbolic algebraic expressions. Another exercise could be a matching game, like "concentration," with the symbolic expression on some cards and corresponding verbal "translations" on others.

Cooperation Between Language Educators and Mathematics Educators. Collaboration between mathematics and language teachers is both beneficial and essential. Language teachers can provide insights into linguistic and cultural problems and offer communicative activities to overcome those problems. The mathematics teacher,

in turn, can suggest appropriate topics for the language courses that reinforce the content the students face in mathematics courses. Admittedly, such collaboration is difficult to establish at the postsecondary level due to the fact that academic departments tend to be more isolated than in K-12 programs. Yet, we have received encouragement from the 1984 and 1988 FIPSE projects, and the New Jersey Institute, as well as from joint projects with college mathematics educators who have demonstrated a sensitivity to the linguistic problem. Our hope is that these efforts will become part of a collaborative blueprint for the future, one which recognizes the need for greater mathematics language proficiency and increased communication in college mathematics courses.

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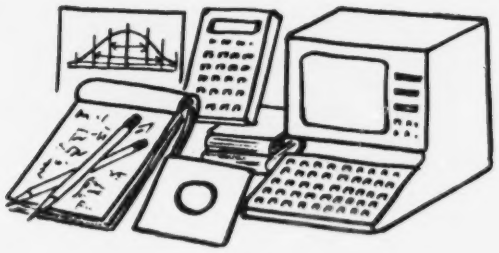
Section 4.
Technology



Mathematical Tools

- Symbolic Computer Systems and the Calculus – by Richard J. Shumway

  "Symbolic computer systems can have a major impact on calculus. Redirection of calculus from an emphasis on procedural knowledge to an emphasis on conceptual knowledge is supported by both informed opinion and research."

- Computer Algebra Systems Approach to Teaching Taylor Polynomials – by John H. Mathews

  "We have seen how shifting the burden of computing to CAS leaves more time to explore meaningful problems, rather than studying only a few routine exercises."

- Ada for Mathematical Programming – by L. S. Tang

  "Ada is a general-purpose programming language developed by the United States Department of Defense. It embodies many advanced software development principles, and is designed to meet the mathematical needs of the 1990s."

Instructional Aids

- Mathematical Multimedia Presentations with Current Technology – by Edward D. Laughbaum

  "Presentation software...can transform an ordinary mathematics lecture into a dynamic multimedia presentation."

- Using Videotapes for Mathematics Instruction – by Karen T. Sharp

  "The growing number of nontraditional students who are enrolling in two-year college programs have nontraditional needs.... One of their most important needs is an instructional delivery system which provides for flexibility.... In my opinion, videotapes and other forms of prerecorded materials can provide this flexibility."

Symbolic Computer Systems and the Calculus

Richard J. Shumway

The Ohio State University, Columbus, Ohio 43210
Centro de Investigación en Matemáticas, Guanajuato, Gto., México 36000

There have been a number of conferences and reports regarding the state of the calculus (e.g., Douglas, 1986 and Steen, 1987) and subsequent reviews of such reports (e.g., Hamming, 1988 and Moore & Smith, 1987). The consensus is that the calculus is in need of change and part of that change is related to the use of technology to do or learn mathematics (Ellis, 1988). The National Science Foundation (1988) has made a clear commitment to exploring change. The National Research Council (1989) has raised the stakes for calculus with the observation that "*...the unique position of calculus as the gateway from school to college mathematics imposes on it* [the calculus] *a special burden to be attractive, compelling, and intellectually stimulating* [italics added]" (p.52) and firmly introduced the role of the computer with the statement that "*Symbolic computer systems compel fundamental rethinking of what we teach and how we teach* [italics added]" (p.52). In short, calculus courses must be conceptually oriented and use symbolic computer systems. Examination of the research related to symbolic computer systems, concept learning, and the calculus will lead to some tentative conclusions and directions for further work.

Symbolic Computer Systems

One of the first computers, credited to Atanasoff & Berry and developed from 1939-1942, was designed to solve linear systems of up to 29 variables (Macintosh, 1988). If not in earlier efforts, by the late 1940s or early 1950s it was realized that computers could be symbol manipulators (Hamming, 1980). Givens (1966) argued the computer would have a major impact on what is thought to be important in mathematics. Kemeny (1966) argued computer programming was an effective strategy for learning mathematical concepts. Stenberg, et al. (1968) had already implemented a computer-based calculus presentation. Birkhoff (1972) shortly thereafter called for calculus with computers including numerical computations, interactive graphing, and formula manipulation. The capabilities of various symbol computer systems are documented in several publications available to teachers of calculus (e.g., Caviness, 1986; Foster & Bau, 1989; Seiter, 1988; Steinberg, 1988; and Wolfram, 1988). Essentially, such systems provide substantial symbolic computation, interactive graphics, and mathematically attractive programming environments.

Doubts about the appropriateness of using symbolic computer systems for mathematics may be relieved by various testimonials from practicing mathematicians (e.g., Appel & Haken, 1981; Cheatham, 1974; Cohen, 1988; Gonnet, 1988; Lambe, 1989; Odlysko, 1985; Pavelle, 1985; Sloane, 1986; Steen, 1985; Wos, 1985; and Young, 1988). There are, of course, respected mathematicians such as Halmos (Albers, 1982), P. Davis (Davis, 1985), and P. Seymour (Kolata, 1984) who raise serious questions about the role for computers in doing mathematics. Perhaps these views are designed to balance those of federal legislators who apparently believe computers will soon make mathematics unnecessary (Murphy, 1988)!

Beyond the obvious interactive graphics and symbolic computational power of such systems, many argue the design and study of computer algorithms is fundamentally mathematical (e.g., Dijkstra, 1974; Friske, 1985; Jaffe, 1984; Karp, 1986; Knuth, 1974, 1985; Leonard & Shultz, 1987; Lovász, 1988; Papert, 1980; and Ralston, 1980). All argue that mathematical proof and algorithm design are best done as coordinated activities. One look at Knuth's books on the art of computer programming (Knuth, 1968, 1969, 1973) or Risch's paper on integration in finite terms (Risch, 1969) convinces most of the fundamental role pure mathematics plays in designing computer algorithms.

Availability of symbolic computer systems is thought to be possible and practical for students through terminals, work stations, and personal computers. Readers will find most published reports of availability not timely enough to be of practical value except as historical reference (e.g., Kraines & Smith, 1988). Contacting colleagues actively using symbolic computer systems is the most practical way to access current information regarding availability (see projects referenced later).

One development worth special mention is the availability of interactive graphics and symbolic

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Dr. Richard Shumway trains mathematics teachers and does research on how mathematics is learned. Author/Editor of books such as 101 Ways to Learn Mathematics Using BASIC (Prentice-Hall) and Research in Mathematics Education (National Council of Teachers of Mathematics), his current interest is the needed, major, conceptual reorganization of school mathematics, K-16.

manipulation on, so called, *supercalculators* (Nievergelt, 1987). The capabilities of such calculators has been described (Muciño, 1988; Tucker, 1987, 1988) and more reference will be made to specific uses in calculus later.

Concept Learning

For educators, Gagné (1965) simply and persuasively described the difference between *concept learning* and other forms of learning through the psychological classification of eight different learning types. Concept learning was the sixth of the eight, coming just before principle learning (e.g., mathematical theorems), and problem solving (e.g., proving new results in mathematics). Summaries of psychological research on concept learning (e.g., Bourne & Dominowski, 1972; Erickson & Jones, 1978; Inhelder & Piaget, 1958) together with educational psychology research (e.g., Tennyson & Park, 1980; Tennyson & Cocchiarella, 1986) have provided some guidelines for researchers in mathematics education. The psychological work confirms the importance of irrelevant attributes, underlying logical operations, examples, nonexamples, and concept prototypes in the learning of concepts and has assisted in making distinctions between *skill learning* (Suydam, 1980) and *concept learning* (Sowder, 1980) in mathematics education. Other terms have been used to make the distinction, i.e., *mechanical* and *meaningful* learning (Brownell & Moser, 1949), or *instrumental* and *relational* learning (Skemp, 1976), but the recent use of *procedural* and *conceptual* knowledge (Hiebert, 1986) seems most helpful, particularly as one thinks about computer applications in mathematics.

Research

There are two fundamental issues firmly intertwined in the minds of most studying the problems of learning calculus. The first is that many calculus students, while computationally proficient, are hopelessly naïve about fundamental concepts of calculus. The second is that, symbolic computer systems make it patently obvious that complicated computations, once thought to distinguish the more mathematically talented students from others, are in fact, routine skills best left to computers. The intertwining nature is the belief that use of symbolic computer systems, with their interactive graphics, symbolic/numeric computations, and mathematical environments will allow (encourage) students to focus more time and attention on concepts and conceptual relationships (theorems).

Conceptual Knowledge

First, the standard horror stories about students with procedural knowledge and little conceptual knowledge (Hiebert, 1986) that involve other levels of mathematics are confirmed at the calculus level too. As well as problems with fundamental concepts such as variable (e.g., Clement, Lockhead, & Soloway, 1980), there are difficulties with the concepts of calculus (Cornu, 1980; Selden, Mason, & Selden, 1989; Shumway, 1988; Tall, 1989; Tufte, 1988). Cornu documents that many preconceived notions about limit seriously interfere with learning a mathematical meaning for limit. Selden, Mason, & Selden report that successful students in calculus are not able to solve nonroutine problems and hypothesize the reason may be that students held a static view of mathematics and that mathematics contained standard procedures to be learned. Shumway found students unable to recognize elementary representations of derivatives in new situations not directly encountered before, a criteria required by Gagné (1965) for evidence of concept learning. He hypothesizes the cause is undue emphasis on computation and inadequate teaching, learning, and testing of conceptual knowledge. Tall notes misconceptions about tangents to curves so strongly held that students modify correct drawings of tangents to incorrect drawings to avoid tangent lines intersecting the curve. Prior misconceptions or incomplete knowledge is hypothesized as the cause. Tufte notes that, after one semester of calculus, 90% could integrate x^2e^x, but only 7% could evaluate the integral of $|x|$ from -1 to 2.

The reasons for successful procedural knowledge coupled with unsuccessful conceptual knowledge may lie with inadequate sampling of the conceptual domain (Tall, 1989; Cornu, 1980; Shumway, 1988) or student beliefs about the nature of knowledge (Selden, Mason, & Selden, 1989). Psychological research supports both possibilities with Tennyson & Park (1980) discussing the expected effects of inadequate sampling of the concept domain and Brabeck (1984) noting three levels of intellectual development in adults, the first which views learning in much the way Selden, Mason, & Selden hypothesized students viewed mathematics.

Symbolic Computer System Approaches

Preliminary, exploratory teaching at several sites has lead many to claim using symbolic computer systems in calculus results in students who are interested in conceptual understanding; experience more varied representations; shift focus from mechanics to central ideas; change perceptions of what is important in mathematics; search for generalizations; examine counterexamples; and develop deeper understanding of the concepts of calculus (e.g., Cipra, 1988; Devitt, 1988; Eidswick, 1988; Flores, 1987; Frantz, 1986; Kemeny, 1988; Kenelly, 1988; Lane, 1985; Lovelock & Newall, 1988; Moon, 1985; Ostebee, 1988; Palmiter, 1988; Sherwood, 1988; Small & Hosack, 1986; Stoutemyer, 1985; Tufte, 1988; and Zorn, 1988).

Heid (1988) has shown that a conceptual emphasis is viable using symbolic computer systems and Palmiter (1986) has shown in a controlled experimental study that using a symbolic computer system one can reduce the time for integral calculus by 50%, improve concept learning, increase computational power, and not debilitate

later required learning of traditional algorithms. While the Palmiter study is currently the only experimental study able to claim potential cause-effect relationships, the multitude of descriptive and exploratory studies cited above should lead to further studies designed to identify cause-effect relationships and develop a model that may explain observed effects.

Studies more narrowly focused on the effects of computer programming on mathematics learning have a longer history and evidence some potential. For example, Clement, Lockhead, & Soloway (1980) have shown programming enhanced student's ability to use variables, Johnson & Harding (1979) have shown a consistent correlation between programming experiences and mathematical problem solving for college mathematics majors. Dubinsky, Elterman, & Gong (1988) have proposed computer programming enhances student understanding of quantified variable and Ayers, Davis, Dubinsky, and Lewin (1988) propose how computer experiences can help students construct mathematical knowledge. Tufte (1988) reports programming experiences improved conceptual understanding of calculus. It would seem appropriate for symbolic computer systems approaches to capitalize on the embedded mathematical programming environments.

The Future

The future can be interpolated from some existing literature. For example, the power of supercalculators is far greater than most believe (e.g., Garner, 1989; Hewlett-Packard, 1988; Mack, 1988; Michel, 1987; Nievergelt, 1987; Shumway, 1989, April; Steen, 1988; Tucker, 1989; and Wickes, 1988). Single keys can be designed for interactive graphics (Tucker, 1989), characteristic equations of matrices, surface areas of revolution, roots of polynomials,polynomial division, derivates, Taylor series, cross and dot products, matrix inverses, or most any other symbolic computation for which one can design a procedure (Wickes, 1988). Such machines have 128K for internal procedures and 32K for user procedures. Their capabilities will increase and, judging from prior history, their cost will likely go down from $165 to $10 (a less dramatic decrease than that of scientific calculators which dropped from $400 to less than $10). The most remarkable impact of the supercalculators is the realization that most exercises in college level mathematics texts can be solved by pressing keys. Clearly such procedures are skills and not concepts.

There is sufficient learning theoretic and mathematical support for counterexamples in the teaching and learning of mathematics (e.g., Tennyson & Park, 1980; and Lakatos, 1976). Computers can support the examination of counterexamples and counterexamples can become a much greater part of the study of mathematics.

Geometric interpretations of mathematical ideas can be explored with much greater facility than before and there are and will be many examples of such representations of calculus concepts (e.g., Bullock, 1988; Dick & Shaughnessey, 1988; Kaput, 1987; Monagan, 1988; and Schoenfeld, 1989).

Fundamental changes in mathematics and mathematics teaching and learning are likely to occur. For example, Cheatham (1974) claims Risch's theory for a finite algorithm for integration will have a profound impact on how mathematics is taught. Demana & Waits (1988) illustrate new ways one can think about graphical representations and their potential misrepresentations. Devitt (1988, July) offers a symbolic computer system proof of the mean value theorem. Dunham (1986) in suggesting a great theorems strategy for organizing a history of mathematics course may be offering an important organizational strategy for the calculus of the future. Mac Lane (1986) offers a model for how mathematics could be thus organized in his book of fundamental ideas of mathematics. Foley (1988) suggests a new organization of calculus based on a blending of classical, discrete and nonstandard analysis. Ferrini-Mundy (1987) provides evidence enhancing spatial visualization of women is possible and will improve achievements in calculus. Perhaps the graphical representations of symbolic computer systems can enhance achievement in calculus. Shumway (1989) argues: a) symbolic computer systems must be available for all teaching, homework, and tests; b) concepts and proofs are now basic mathematics; c) substantial changes in the curriculum must be made; and d) teaching strategies must focus on examples, nonexamples, and proofs.

Hamming (1988) in his review of the lean and lively calculus report, predicts big changes are necessary but doubts mathematicians will respond. Fey (1988) notes that "*While some may choose to wait until a clearer picture of the 'best' response emerges, the situation right now offers impressive opportunities for progress.... There are many questions to be answered but we really have no choice but to tackle these questions and bring school and university mathematics into the electronic information age for which we are ostensibly preparing our students* [italics added]" (p. 236).

These are exciting times that will require all of us to relearn our jobs, not in the traditional every ten years time, but more likely in a new and fast-paced every two years time. Such change only accentuate the idea we are teaching mathematical thinking and not mathematical topics. Learning mathematical thinking rather than performing mathematical procedures is the fundamental goal for students of mathematics.

Summary

Substantial interest has been generated in calculus revision and the use of computers in the teaching of calculus. Symbolic computer systems can have a major impact on calculus. Redirection of calculus from an emphasis on procedural knowledge to an emphasis on conceptual knowledge is supported by both informed opinion and research. Concepts, examples, nonexamples, and proofs must receive major emphasis.

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Computer Algebra Systems Approach to Teaching Taylor Polynomials

John H. Mathews

Department of Mathematics, California State University, Fullerton, CA 92634

Computer algebra systems (CAS's) such as DERIVE, MACSYMA, MAPLE, muMATH and Mathematica can be used as powerful assistants for performing symbol manipulations in algebra and calculus. It has been suggested that these systems will benefit undergraduates in mathematics by keeping track of the details in complicated manipulations (see Cromer, 1987; Freese, Lounesto & Stegenga, 1986; Small & Hosack, 1986; Woof & Hodgkinson, 1987; and Zorn, 1986). This article shows how the symbolic calculus and graphics capabilities of the two CAS systems MAPLE and Mathematica can be used to enrich the understanding of both the theoretical and computational aspects of Taylor series. Both systems are available on Macintosh computers, and to fully utilize their capabilities requires several megabytes of RAM memory. The Mathematics Department at CSUF uses a satellite computer laboratory in our School of Natural Science and Mathematics. It currently includes 19 students work stations.

The Taylor series expansion of a function $f(x)$ is a familiar topic of sophomore calculus. It is the foundation of much theory in analysis and applied mathematics. Unfortunately, it is often viewed by students to be difficult to understand. Using assistance from CAS to construct the coefficients of the series, draw graphs and manipulate polynomials of high degree, gives us more time to focus our attention on the underlying principles. Using MAPLE and Mathematica has generated enthusiasm for both students and teachers.

THEOREM [Taylor]. Let f be infinitely differentiable on an interval containing the numbers x_0 and x. Then, for each positive integer n, $f(x)$ can be represented as the sum

$$(1)\quad f(x) = P_n(x) + R_n(x), \qquad \text{where}$$

$$(2)\quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

$$(3)\quad R_n(x) = \frac{f^{(n+1)}(c)(x-x_0)^{n+1}}{(n+1)!}$$

where $c = c(x)$ lies somewhere between x_0 and x.

The MAPLE syntax for initiating the computation of a Taylor series is

```
taylor(f, x =x0, n + 1);
```

which returns the n-th degree Taylor series for f with respect to the variable x expanded about the value x_0. For example, the Maclaurin series of order $n=7$ for $\sin(x)$, $\cos(x)$, $\exp(x)$ and $\ln(1+x)$ are found by issuing the commands:

```
taylor(sin(x), x =0, 8);
```

$$x - 1/6\,x^3 + 1/120\,x^5 - 1/5040\,x^7 + O(x^8)$$

```
taylor(cos(x), x =0, 8);
```

$$1 - 1/2\,x^2 + 1/24\,x^4 - 1/720\,x^6 + O(x^8)$$

```
taylor(exp(x), x =0, 8);
```

$$1 + x + 1/2\,x^2 + 1/6\,x^3 + 1/24\,x^4 + 1/120\,x^5 + 1/720\,x^6 + 1/5040\,x^7 + O(x^8)$$

```
taylor(log(1+x), x =0, 8);
```

$$x - 1/2x^2 + 1/3x^3 - 1/4\,x^4 + 1/5\,x^5 - 1/6\,x^6 + 1/7x^7 + O(x^8)$$

MAPLE has included the terms $O(x^8)$, which means that the coefficients of the Taylor series have been computed up to x^7 and that an error term must be included and it involves several factors, one of which is x^8. In order to do computations with the Taylor series, we must use MAPLE's command to convert them to polynomial form, i.e. we must dig out the polynomial part. For example, suppose that we want to view the polynomial approximations of degree $n=3,5,7,9$ to $\sin(x)$ over the interval $[-2\pi, 2\pi]$. Then we issue the MAPLE commands:

```
p3 := convert(taylor(sin(x), x=0, 4),polynom);
```

$$p3 := x - 1/6\,x^3$$

John earned his PhD in mathematics from Michigan State University under the direction of Peter Lappan. He has been a member of the mathematics faculty of California State University, Fullerton, for twenty years, and keeps active in the areas of complex analysis, numerical analysis and computer algebra.

```
p5 := convert(taylor(sin(x), x=0, 6),polynom);
```

$$p5 := x - 1/6\,x^3 + 1/120\,x^5$$

```
p7 := convert(taylor(sin(x), x=0, 8),polynom);
```

$$p7 := x - 1/6\,x^3 + 1/120\,x^5 - 1/5040\,x^7$$

```
p9 := convert(taylor(sin(x), x=0, 10),polynom);
```

$$p9 := x - 1/6\,x^3 + 1/120\,x^5 - 1/5040\,x^7 + 1/362880\,x^9$$

We are now ready to look at the four polynomial approximations $P_3(x)$, $P_5(x)$, $P_7(x)$ and $P_9(x)$ to $f(x) = \sin(x)$ (see Figure 1).

```
plot({sin(x),p3,p5,p7,p9},-2*Pi..2*Pi,-2..2);
```

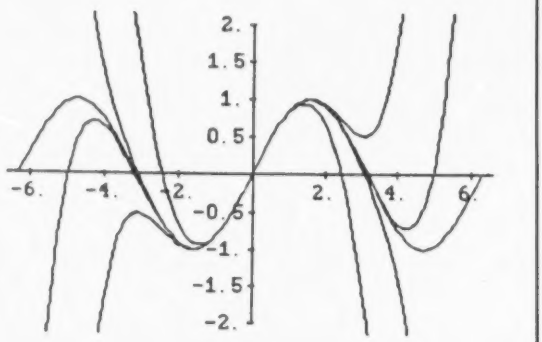

Figure 1

Looking at Figure 1 reveals that the approximations of higher degree track the curve $y = \sin(x)$ closer over a wider interval. Further investigations can be made by the students.

The details for expanding f in a Taylor series about $x_0 \neq 0$ usually requires more work than a Maclaurin series and is error prone. Let us switch to Mathematica and continue our CAS investigations. We expand $\sin(x)$ in a Taylor series of degree $n = 5$ about $x_0 = \pi/3$ by issuing the Mathematica command:

```
t5 = Series[Sin[x],{x,Pi/3,5}]
```

$$\frac{\text{Sqrt[3]}}{2} + \frac{\frac{-\text{Pi}}{3}+x}{2} - \frac{\text{Sqrt[3]}\left(\frac{-\text{Pi}}{3}+x\right)^2}{4} - \frac{\left(\frac{-\text{Pi}}{3}+x\right)^3}{12} + \frac{\text{Sqrt[3]}\left(\frac{-\text{Pi}}{3}+x\right)^4}{48} + \frac{\left(\frac{-\text{Pi}}{3}+x\right)^5}{240} + 0\left[\frac{-\text{Pi}}{3}+x\right]^6$$

Notice that Mathematica has used the notation Pi and Sqrt[3] for π and $3^{1/2}$, respectively. Notational conventions such as these are typical for CAS and it is best to stare at the output until it becomes familiar. After we convert the Taylor series to a polynomial, we can graph it and see that it produces an approximation that is not symmetric about the origin (see Figure 2).

```
p5 = Normal[t5]
plot({sin[x],p5},{x,-Pi, 3 Pi/2}]
```

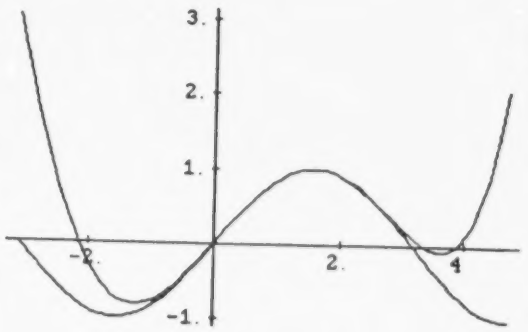

Figure 2

Students immediately find that using CAS is a powerful tool for doing homework problems. However, in order to understand the theory they must learn how the coefficients and error term are derived. This too can be assisted with Mathematica, but it is necessary to write a short program that prints out the intermediate steps. The syntax for this new Mathematica procedure is

```
Taylor[f, x0, n];
```

and the listing is given in the appendix at the end of the article. It produces more information and helps students understand the underlying principles when the details are messy.

EXAMPLE 1. Find the Taylor polynomial of degree $n = 8$ and the corresponding error term for the function $f(x) = \exp(-x)\sin(x)$ expanded about the value $x_0 = 0$.

SOLUTION. First, we can place $\sin(x)$ in the variable F by typing:

```
F = Exp[-x] Sin[x]
```

and the computer responds

$$\frac{\text{Sin}[x]}{E^x}$$

Issue the new Mathematica command:

```
Taylor[F, 0, 8];
```

Then Mathematica will print the following information:

Formulas for the derivatives:	Numerical values for the derivatives:
$f^{(0)}[x] = \frac{\text{Sin}[x]}{E^x}$	$f^{(0)}[0] = 0$
$f^{(1)}[x] = \frac{\text{Cos}[x]}{E^x} - \frac{\text{Sin}[x]}{E^x}$	$f^{(1)}[0] = 1$
$f^{(2)}[x] = \frac{-2\,\text{Cos}[x]}{E^x}$	$f^{(2)}[0] = -2$
$f^{(3)}[x] = \frac{2\,\text{Cos}[x]}{E^x} + \frac{2\,\text{Sin}[x]}{E^x}$	$f^{(3)}[0] = 2$
$f^{(4)}[x] = \frac{-4\,\text{Sin}[x]}{E^x}$	$f^{(4)}[0] = 0$
$f^{(5)}[x] = \frac{-4\,\text{Cos}[x]}{E^x} + \frac{4\,\text{Sin}[x]}{E^x}$	$f^{(5)}[0] = -4$
$f^{(6)}[x] = \frac{8\,\text{Cos}[x]}{E^x}$	$f^{(6)}[0] = 8$
$f^{(7)}[x] = \frac{-8\,\text{Cos}[x]}{E^x} - \frac{8\,\text{Sin}[x]}{E^x}$	$f^{(7)}[0] = -8$
$f^{(8)}[x] = \frac{16\,\text{Sin}[x]}{E^x}$	$f^{(8)}[0] = 0$

The function is: $F[x] = \frac{\text{Sin}[x]}{E^x}$

The Taylor polynomial is:

$$P_8[x] = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} - \frac{x^7}{630}$$

The Lagrange form of the remainder term is:

$$R_8[x] = \frac{x^9\left(\frac{16\,\text{Cos}[c]}{E^c} - \frac{16\,\text{Sin}[c]}{E^c}\right)}{362880}$$

where c lies somewhere between 0 and x.

In this example, the pattern for the derivatives appears to involve multiples of the sequence $\{0, 1, -2, 2\}$, a fact that can now be proven by induction.

EXAMPLE 2. Find the Taylor polynomial of degree n = 9 and the corresponding error term for the function $f(x) = x^{1/3}$ expanded about the value $x_0 = 8$.

SOLUTION. First, we can place $x^{1/3}$ in the variable F with the command:

```
F = x^(1/3)
```

and the computer's response is

$$x^{\frac{1}{3}}$$

Then issue the Mathematica command:

```
Taylor[F, 8, 9];
```

Again, a complete list of formulas and values of the derivatives will be printed by Mathematica. For brevity, we have edited this printout.

Formulas for the derivatives:	Numerical values for the derivatives:
$f^{(1)}[x] = \frac{1}{3x^{\frac{2}{3}}}$	$f^{(1)}[8] = \frac{1}{12}$
$f^{(2)}[x] = \frac{-2}{9x^{\frac{5}{3}}}$	$f^{(2)}[8] = -\left(\frac{1}{144}\right)$
$f^{(3)}[x] = \frac{10}{27x^{\frac{8}{3}}}$	$f^{(3)}[8] = \frac{5}{3456}$

The Taylor polynomial is:

$$P_9[x] = 2 + \frac{-8+x}{12} - \frac{(-8+x)^2}{288} + \frac{5(-8+x)^3}{20736}$$

$$- \frac{5(-8+x)^4}{248832} + \frac{11(-8+x)^5}{5971968} - \frac{77(-8+x)^6}{429981696}$$

$$+ \frac{187(-8+x)^7}{10319560704} - \frac{935(-8+x)^8}{495338913792} + \frac{21505(-8+x)^9}{106993205379072}$$

The Lagrange form of the remainder term is:

$$R_9[x] = \frac{-55913x^{10}}{4782969c^{\frac{29}{3}}}$$

where c lies somewhere between 8 and x.

Since $x^{1/3}$ is not symmetric about $x_0 = 8$, it is interesting to view this approximation. For convenience, we use the interval [0,27] (see Figure 3).

plot[{F,S},{x, 0, 27}]

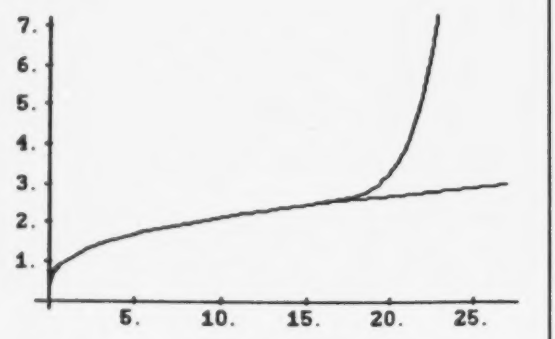

Figure 3.

Interval of Convergence.

Another important topic about infinite series is the interval of convergence. For illustrative purposes, we shall consider the Runge function $f(x) = 1(x^2 + 1)$.

```
f = 1/(x^2 + 1)
```

$$\frac{1}{1+x^2}$$

The Taylor polynomials of degree n = 10 and n = 12 expanded about x = 0 are obtained when we issue the commands:

```
p10 = Normal[Series[f, {x, 0, 11 }]]
```

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10}$$

and

```
p12 = Normal[Series[f, {x, 0, 13 }]]
```

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12}$$

We can visually inspect how close these approximations are to $f(x)$ when we plot the graphs of f, $P_{10}(x)$ and $P_{12}(x)$ (see Figure 4).

plot[{f, p10, p12},{x,-1.5, 1.5}]

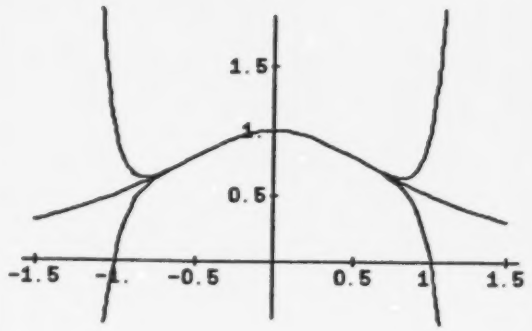

Figure 4.

Why do the approximations P_{10} and P_{12} diverge away from f near $x = 1$ and $x = -1$? Because, $P_{10}(1) = P_{10}(-1) = 0$, $P_{12}(1) = P_{12}(-1) = 1$ and $f(1) = f(-1) = 1/2$. The error for all of these cases is $\pm 1/2$. This pattern is continued with all of the partial sums of the infinite Taylor series. The Taylor series will converge to $f(x)$ when $-1 < x < 1$ but will not converge at the endpoints $x = \pm 1$. The theoretical approach is to investigate the error term $E_{10}(x) = f^{(11)}(c)x^{11}/11!$ We can gain insights after we plot the function $g(x) = f^{(11)}(x)/11!$ (see Figure 5).

```
g = Together [ (D[f, {x, 11}])/11!]
```

$$\frac{12x - 220x^3 + 792x^5 - 792x^7 + 220x^9 - 12x^{11}}{(1 + x^2)^{12}}$$

```
plot[g, {x, -1.5, 1.5}]
```

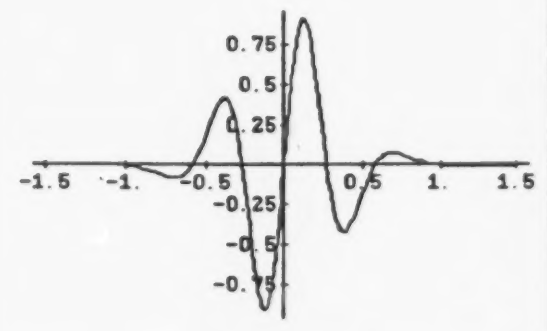

Figure 5.

Looking at the graph of $g(x)$ we can see that $|g(x)| < 0.95$ and since $|x^{11}| < 1$ when $|x| < 1$, we have established the error bound

$$|E_{10}(x)| = |g(c)|\ |x^{11}| \le 0.95 * 1 = 0.95.$$

Investigations such as these take some of the mystery out of mathematical analysis.

The Calculus of Power Series

The study of power series should be viewed as the extension of the study of polynomials. The connection can be made when we use the symbol manipulation power of Mathematica to investigate meaningful examples.

```
cosine = Normal[Series[Cos[x],{x, 0, 13}]]
```

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600}$$

The termwise derivative of this polynomial is easily found:

```
D[cosine, x]
```

$$-x + \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040} - \frac{x^9}{362880} + \frac{x^{11}}{39916800}$$

This can be compared with the series for sin x.

```
sine = Normal [Series [Sin[x] , {x, 0, 12}]]
```

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800}$$

The conclusion to be drawn is that d/dx $\cos(x) = -\sin(x)$. A well known identity from trigonometry can be attempted when we issue the command

```
Normal[sine^2 + cosine^2]
```

$$1 - \frac{x^{14}}{3353011200} + \frac{x^{16}}{22992076800} - \frac{x^{18}}{517321728000} + \frac{x^{20}}{24141680640000} - \frac{x^{22}}{1912021106688000} + \frac{x^{24}}{229442532802560000}$$

The interpretation to be made is that this is a good approximation to 1 when x is small or even when $|x| < 1$. This can be contrasted with the series manipulation approach given in Example 3 below, but be prepared to answer some sticky questions.

The Cauchy Product

CAS can be usedto illustrate abstract ideals by finding the Taylor series of the product $f(x)g(x)$. It is known that the formula involves a linear combination of derivates and binomial coefficients. First, we can discover the pattern for the higher derivates of the product $h(x) = f(x)g(x)$ with the following dialogue.

```
D[f[x] g[x], x]
```

$$g[x]\, f'[x] + f[x]\, g'[x]$$

The second derivative of the product is:

```
D[f[x] g[x] , {x, 2}]
```

$$2\, f'[x]\, g'[x] + g[x]\, f''[x] + f[x]\, g''[x]$$

Similarly, the third and fourth derivatives are:

```
D[f[x] g[x] , {x, 3}]
```

$$3\, g'[x]\, f''[x] + 3\, f'[x]\, g''[x] + g[x]\, f^{(3)}[x] + f[x]\, g^{(3)}[x]$$

and

```
D[f[x] g[x] , {x, 4}]
```

$$6\, f''[x] g''[x] + 4 g'[x] f^{(3)}[x] + 4 f'[x] g^{(3)}[x] + g[x]\, f^{(4)}[x] + f[x]\, g^{(4)}[x]$$

A regrettable feature of CAS is that the answer might not appear in a format that the user expects or wants; i.e., the previous results are illuminated when they are rewritten as

$$f g' + f' g$$
$$f g'' + 2 f' g' + f'' g$$
$$f g''' + 3 f' g'' + 3 f'' g' + f''' g$$
$$f g'''' + 4 f' g''' + 6 f'' g'' + 4 f''' g' + f'''' g$$

The pattern of binomial coefficients can now be recognized. Therefore, our Mathematica experimentation led us to the conjecture

$$h^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(x) g^{(k-j)}(x),$$

which can be proven with mathematical induction. Therefore, the Taylor polynomial $P_n(x)$ of degree n for the product $h(x)$ is

$$(4)\quad P_n(x) = \sum_{k=0}^{n} \frac{h^{(k)}(x_0)}{k!}(x-x_0)^k,$$

$$(5)\quad h^{(k)}(x_0) = \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(x_0) g^{(k-j)}(x_0).$$

EXAMPLE 3. Verify that formulas (4) and (5) can be used to find the Taylor polynomial of degree n = 4 for the function $h(x) = \exp(x)\cos(x)$.

SOLUTION. Let $f(x) = \exp(x)$ and $g(x) = \cos(x)$, then we know that

$f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1$ and $f''''(0) = 1$

$g(0) = 1, g'(0) = 0, g''(0) = -1, g'''(0) = 0$ and $g''''(0) = 1$

Therefore,

$$h(0) = 1(1)(1) = 1$$

$$h'(0) = 1(1)(0) + 1(1)(1) = 1$$

$$h''(0) = 1(1)(-1) + 2(1)(0) + 1(1)(1) = 0$$

$$h'''(0) = 1(1)(0) + 3(1)(-1) + 3(1)(0) + 1(1)(1) = -2$$

$$h''''(0) = 1(1)(1) + 4(1)(0) + 6(1)(-1) + 4(1)(0) + 1(1)(1) = -4$$

Hence, the desired Taylor polynomial is

$$P_4(x) = 1 + 1/1!\,x + 0/2!\,x^2 - 2/3!\,x^3 - 4/4!\,x^4$$
$$= 1 + x - x^3/3 - x^4/6$$

Of course, this can be verified with the Mathematica

```
cosx = Series[Cos[x] , {x, 0, 4 }]
```

$$1 - \frac{x^2}{2} + \frac{x^4}{24} + O[x]^5$$

```
expx = Series[Exp[x] , {x, 0, 4 }]
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O[x]^5$$

```
cosx expx
```

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + O[x]^5$$

Notice that Mathematica has multiplied the two series together and ignored terms involving powers of x larger than 4. After all, this is to be expected, because if you do not yet know the coefficients for x^5 in the series for $\cos(x)$ and $\exp(x)$ then you cannot compute it in their product.

Conclusion

We have seen how shifting the burden of computing to CAS leaves more time to explore more meaningful problems, rather than studying only a few routine exercises. Details involving error analysis can be obtained for several complicated functions. Graphs of functions and their higher derivatives are easy to obtain. The obstacle of not enough time is erased and implementing an experimental approach is feasible. It does open up the door to asking questions, many of which cannot be answered on the spot. If an attitude of encouragement is used then the situation makes students active participants in mathematical exploration, rather than passive recipients of a fixed body of facts.

Appendix

The Mathematica program listing which prints out the details for a Taylor series is:

```
Taylor[f_,x0_,n_] := {Print ["Formulas for the derivatives:"],
                      Print[" "],
                      Do[{Der = D[f, {x, k}],
                          Print["f",Superscript["(",k,")"],
                              "[x] = ", Der], Print[" "]},
                         {k, 0, n}],
                      Print["Numerical values for the derivatives:"],
                      Print[" "],
                      Do[{Der = D[f, {x, k}],
                          Val = ReplaceAll[Der,x→x0],
                          Print["f",Superscript["(",k,")"],
                              "[",x0,"] = ", Val], Print[" "]},
                         {k, 0, n}],
                      Print["The function is:","\n"," F[x] = ",f],
                      Print[" "],
                      Print["The Taylor polynomial is:"],
                      Print[" "],
                      S = Normal[Series[f, {x, x0, n}]],
                      Print["P",Subscript[n],"[x] = ", S],
                      Print[" "],
                      m = n + 1,
                      Der = D[f, {x, m}],
                      Err = ReplaceAll[Der,x→c]^m/m!,
                       Print["The Lagrange form of the remainder
                         term is:"],
                      Print[" "],
                      Print["R",Subscript[n],"[x] = ", Err],
                      Print[" "],
                      Print["where c lies somewhere between ",
                           x0," and x."] }
```

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Ada® for Mathematical Programming

L. S. Tang

Western New England College, Springfield, MA 01119

Ada is a general-purpose programming language developed by the United States Department of Defense. It embodies many advanced software development principles, and is designed to meet the mathematical needs of the 1990s. Ada is completely defined and standardized. That means that a compiler that implements a subset or a superset of Ada will not be validated. Therefore, an Ada program written for one computer system can also run on the other systems. Furthermore, Ada has extensive features for controlling arithmetic precision, which makes it very suitable for numerical computation.

Ada supports both top-down and bottom-up designs. The top-down methodology is appropriate for programs that can be broken down into logical subunits. We can develop programs without immediate concern about the details of subunits. The bottom-up design is used for the development of libraries for general use. Ada supports this through the use of **packages**. Packages are compiled separately and are easily accessible to all application programs. Other languages, such as FORTRAN, may also support separate compilation, but the order of compilations and parameter passing are not enforced by the compiler. Therefore, a library procedure may be called with the wrong number of parameters, or with parameters in the wrong order. With Ada, the compiler manages a data base so that it can keep track of the logical dependencies between separately compiled components. Therefore, Ada is more appropriate for the development of mathematical libraries.

Ada also supports generic program units. A generic unit is a template from which we can create many executable versions; each version can be used for a special type of data. For example, a library of set operations can be written generically, and used for sets of integers, sets of characters, etc.

Ada supports parallel processing. That is, several segments (called **tasks**) of a program can run simultaneously on a multiprocessor computer. Mathematical problems often have many phases of computation that are independent of each other, and hence may be computed simultaneously to reduce execution time.

In this paper I will discuss these powerful features of Ada, and how they may be used for mathematical programming.

Arithmetic Precisions

Ada allows the programmer to define the required arithmetic precision. For example, if in certain computation we need an accuracy of 10 significant digits, we can define a numerical type as follows:

```
type VALUE is digit 10;
```

This means that VALUE is a **floating point** type with a relative accuracy of at least ten digits. The maximum number of digits of accuracy that can be specified is system dependent.

A numerical type in Ada may also be declared to have an absolute accuracy, called **fixed point** representation. A fixed point type is defined by a declaration such as

```
type VALUE is delta 0.0001;
```

In this case VALUE will have an absolute accuracy of at least 0.0001.

Packages

Ada's most important feature is the **package**. A package is a collection of data declarations, and operations for manipulating these data types. Packages are separately compiled, and easily accessible to all application programs. Therefore, general-purpose routines such as numerical integrations, complex number operations and vector operations can be written in packages.

A package in Ada usually consists of two parts: a **specification** and a **body**. The specification specifies what facilities (such as data types, variables, and subprograms) are available to the user programs. Since the application programs need not, and should not, know how the subprograms are implemented, the details of implementation (the actual code) are "hidden" in the body of the package.

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L. S. Tang received his PhD in Mathematics in 1977 from Rutgers University, and his MS in Computer Science in 1984 from Rensselaer Polytechnic Institute. His research interests include programming languages, theory of computation and computer algorithms.

Example 1 — A package of vector operations

The following package defines two operations, addition and dot product, for vectors. Notice that in Ada we can define operators using mathematical notations "+", "−", "*", "/", "<", etc. This is called **overloading**. It makes Ada programs more understandable. Compilers will be able to distinguish the vector operations we defined in this package from the predefined real and integer operators by the context provided by the types of variables in the parameter list.

```
package Vector_Operations is
   -- package specification
   -- Here we define a vector type and 2 operations
   type Vector is array(1 .. 10) of Integer;
   function "+" (A, B: Vector) return Vector;
   function "*" (A, B: Vector) return Integer;
end Vector_Operations;
   -- end of package specification

package body Vector_Operations is -- package body
   --It contains the actual code of subprograms
   -- declared in the package specification.
   function "+" (A, B: Vector) return Vector is
      I : Integer;
      C : Vector; -- C is the sum of A and B
   begin
      for I in 1 .. 10 loop
         C(I) := A(I) + B(I);
      end loop;
      return C;
   end "+";
   function "*" (A, B : Vector) return Integer is
      I : Integer;
      D : Integer; -- D is the dot product of A and B
   begin
      D := 0;
      for I in 1 .. 10 loop
         D := D + A(I) * B(I);
      end loop;
      return D;
   end "*";
end Vector_Operations; -- end of package body
```

Once the package is compiled, we can use it in any application program simply by adding the **with** and **use** clauses at the beginning of the program as shown in the following example:

```
with Vector_Operations;
use Vector_Operations;
   -- These two clauses tell the compiler
   -- that we will use the Vector_Operations
   -- package in this program. The compiler
   --will retrieve from its data base the
   -- information describing the interface
   -- of Vector_Operations so that it can
   -- check that the package is used correctly,
   -- and link the package to the program
   -- at link time.

procedure Application is
   A, B, X : Vector;
   Dot_Product : Integer;
begin
   ...
   X := A + B; -- find sum of A and B
   Dot_Product := A * B;
   -- find dot product of A and B
   ...
end Application;
```

Generic Subprograms and Packages

Ada is a strongly typed language. Therefore, a subprogram can be called only if parameters match exactly in number and types. However, in many situations this limitation is not appropriate. For example, when we write a package of set operations, we would like it to work with various types of sets, such as sets of integers and sets of characters. Ada solves this problem by using **generic subprograms** and **generic packages**.

A generic subprogram can operate on more than one data type. Such subprograms can have types, and other subprograms as their parameters. These are called **generic formal parameters**. We can consider a generic formal parameter as a template. Its actual value will be supplied when we use the subprogram. The process of substituting the actual values of generic formal parameters and creating an executable version of the subprogram is called **generic instantiation.**

Example 2 — A generic set package

Ada does not have predefined set operations. We can construct a generic set package as follows:

```
generic
   type Elements is (< >);
      -- This is a generic formal parameter.
      -- (< >) will match any enumeration type,
      -- such as Integer and Character.
packageSet_Operation is -- package specification
   type Set is array(Elements) of Boolean;
   -- a is in set S iff S(a) is true
   function "+" (S1, S2 : Set) return Set;
   -- union of two sets
   function "*" (S1, S2 : Set) return Set;
   -- intersection of two sets
   ... -- declaration of other set operations,
      -- such as "-" (difference),
      -- "<" (subset), complement, insert, delete, etc.
end Set_Operation; -- end of package specification
package body Set_Operation is -- package body
   function "+" (S1, S2 : Set) return Set is
   begin
      return (S1 or S2);
```

```
      -- In Ada we can "or" two boolean arrays
   end "+";
   ... -- implementation of other set operations
end Set_Operation; -- end of package body
```

The following program shows how to use this generic package:

```
with Set_Operation;
procedure Application is
   package Int_Set is new Set_Operation (Integer);
      -- This is generic instantiation.
      -- Here we substitute Integer for
      -- the generic formal parameter "Elements".
      -- We now have a package
      -- for manipulating sets of integers.
   use Int_Set;
   S, S1, S2 : Set;
begin
   ...
   S := S1 + S2; -- S is the union of S1 and S2
   ...
   ...
end Application;
```

Example 3 — A generic integration subprogram

When we write a numerical integration subprogram, it should work for many different functions. But, Ada does not normally allow functions to be passed as parameters to a subprogram. Therefore, we have to use functions as generic formal parameters. The following is an integration routine written as a generic subprogram:

```
generic
   with function F (X : Float) return Float;
      -- Here the function F is
      -- a generic formal parameter.
function Simpson (A, B : Float; N : Integer) return Float;
      -- This is generic function specification.
      -- It will find an approximation
      -- of the integral of f(x) over [a,b]
      -- using the Simpson's rule with
      -- n subintervals.
function Simpson (A, B : Float; N : Integer)
      return Float is -- function body
   ... -- declaration of local variables
begin
   ... -- code for the Simpson's rule
end Simpson;
```

The following example shows how to use this integration subprogram:

```
with Simpson;
procedure Application is
   function My_Function (X : Float) return Float is
   begin
      return (X * X + 1)/(X ** 5 + 2);
   end My_Function;
   function Integral is new Simpson(My_Function);
      -- This is generic instantiation.
      -- The generic formal parameter
      -- F is now replaced by My_Function.
   Value : Float;
begin
   Value := Integral(1.0, 4.0, 20);
      -- approximate the integral of My-Function
      -- over [1, 4] using 20 subintervals.
   ...
end Application;
```

Tasks for Parallel Processing

Computers with several cpu processors can perform many different operations simultaneously. This is called **concurrent** or **parallel processing**. As computer hardware becomes more advanced, parallel processing will become more common. However, most current languages only support sequential processing; that is, a program can only execute one statement at a time. Therefore, these languages will not be able to use the parallel processing capability of future computers.

Ada allows parallel processing by using **Tasks**. Tasks are program units whose execution can proceed in parallel. The main program itself is a task. Tasks can interact with each other by a mechanism called **rendezvous**. A rendezvous between two tasks occurs when one task (the **user task**) calls an **entry** declared in the other task (the **server task**). Data can be passed in and retrieved from a server task through the parameters of an entry point. For each entry declared in the task specification, there is a corresponding **accept** statement in the task body. The two tasks will start in parallel and execute down to the rendezvous point. The task that reaches the rendezvous first will wait for the other task. When they are both at the rendezvous, instructions in the accept statement will be executed, and the user task will wait at that point. After all instructions in the accept statement are completed, the two tasks will then continue in parallel.

Example 4 — Matrix product

Consider the matrix multiplication AB, where A is an $n \times n$ matrix and B is a column vector of n elements. Using sequential processing, we have to calculate the dot product for each row of A with B; one after the other. Using the parallel processing, we can perform these n dot products simultaneously. We will declare an array of n tasks to do these n dot products. We first pass B and the i^{th} row of A to the i^{th} task through one entry point, then retrieve the result of dot product through the other entry point.

```
procedure Matrix_Product is
   type Vector is array (1..N) of Integer;
   type Matrix is array (1..N) of Vector;
```

```
   task type A_Dot_Product is
      entry Pass_Vectors (V1, V2 : in Vector);
      entry Get_Result (X : out Integer);
   end A_Dot_Product;
   Dot_Products : array (1..N) of A_Dot_product;
   A : Matrix;
   B, Product : Vector;
   task body A_Dot_Product is -- task body
      X, Y : Vector;
      Result : Integer;
   begin
      accept Pass_vectors (V1, V2 : in Vector)
         do -- rendezvous point
            X := V1;
            Y := V2;
      end Pass_vectors; -- end of rendezvous
      ... -- calculate the dot product of X
         -- and Y and store the result
      ... -- in the variable Result
      accept Get_Result (X : out Integer) do
         -- rendezvous point
         X := Result;
      end Get_Result; -- end of rendezvous
   end A_Dot_Product;
   ...
begin -- main program (the user task in this example)
   ...
   for I in 1 .. N loop
      Dot_Products(I).Pass_Vectors(A(I), B);
         -- rendezvous by calling an entry point
         -- of the server task
   end loop;
   for I in 1..N loop
      Dot_Products(I).Get_Result(Product(I));
         -- rendezvous by calling another entry point
   end loop;
   ...
end Matrix_Product;
```

◇ ◇ ◇

Conclusion

It is my firm belief that Ada is an excellent language for mathematical programming. We can write mathematical routines using packages, generic subprograms and packages. It is especially suitable for numerical computation because of its facilities for controlling arithmetic precision. Tasks can reduce computation time by taking full advantage of parallel processing capability of modern computers. Its powerful features make mathematical programming easier and more efficient.

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Multimedia Mathematical Presentations With Current Technology

Edward D. Laughbaum

Columbus State Community College, Columbus, Ohio 43215

Presentation software, such as PC Storyboard, Storyboard Plus, Show Partner, or Concorde (see Appendix for more information) can transform an ordinary mathematics lecture into a dynamic multimedia presentation. It can be used for something as simple as teaching operations with signed numbers or as complex as the fundamental theorem of calculus.

Using Presentation Software

To use presentation software, the instructor prepares a lecture in the usual way, except he enters his notes on a monitor and saves each screen to a disk file. Then, instead of using a blackboard or overhead projector to display the content, he uses a computer and projection system. The software package enables the instructor to fill the monitor with mathematical expressions, equations, and graphs that are projected to a screen. Expressions can be written in vivid colors with bold, italic and/or shadowed styles, while different-sized text and a variety of fonts add visual interest. As a reinforcement tool, boxes or ellipses can be drawn to highlight answers or important points. The instructor can also capture screens from other software and edit them as needed.

For example, when using the distributive property, the instructor can make moving arrows appear on the screen, pointing the way to the next step of the algebraic process (see Figure 1). When graphing a linear function, the instructor can make the x-axis move in from the left or right, the y-axis move in from the top or bottom – all in a matter of one or two seconds. The graph of the function can be made to move in from any appropriate direction; that is, it starts as a point and grows to its full length. Another method of display is to start with just the y-intercept being graphed, followed by a line moving up (rise) and a line moving right (run). Next a second point will appear on the graph. This can be done for several points and then the entire graph can be made to fade-in point-by-point (see Figure 2). If the instructor wants to show the approximate area under a curve, he can first display the curve. Students can watch as each rectangle under the curve starts at the x-axis and moves up to the curve. The instructor can control the number, color, and speed of rise of each rectangle (see Figure 3). When teaching division of polynomials, in step four of the division algorithm, the next term can literally move down to the proper place (see Figure 4).

The following screen dumps of the above examples demonstrate what the screens look like as the instructor creates them. The student sees them displayed as described in the above text.

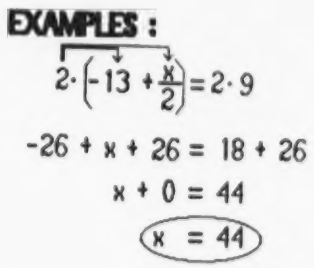

Figure 1. Showing the algebraic process

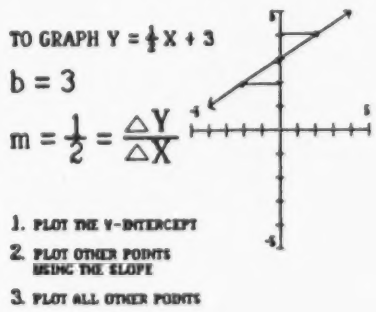

Figure 2. Graphing a linear function

Ed has been in the Mathematics Department at Columbus State for 18 years. He received his undergraduate degree from Otterbein College and his MA from The Ohio State University.

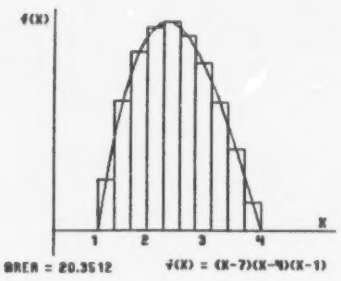

Figure 3. Finding the area under a curve

$$
\begin{array}{r}
4X^2 - 13X + 28 \\
X + 2 \overline{)\,4X^3 - 5X^2 + 2X + 8} \\
-(4X^3 + 8X^2) \\
\hline
-13X^2 + 2X \\
-(-13X^2 - 26X) \\
\hline
28X + 8 \\
-(28X + 56) \\
\hline
-48
\end{array}
$$

1 DIVIDE FIRST TERM BY FIRST TERM
2 MULT. PARTIAL QUOTIENT TIMES DIVISOR
3 SUBTRACT
4 BRING DOWN NEXT TERM THEN GO TO STEP ONE

Figure 4. The division algorithm

The instructor is always in control of what is said and what is displayed. However, by using presentation software, *it is possible and it is simple* to put on a multimedia presentation that will give students an experience totally different from what they have ever had before. As the material is brought to the monitor, the lecture is ongoing. The instructor's part of the presentation is to add voice and orchestrate all the parts that constitute a multimedia presentation. For example, the board may be used in conjunction with what is displayed on the monitor; the instructor might pose a problem on the board and then solve it on the screen. Or the instructor may interrupt the presentation and let the students work a problem given on a hand-out.

Student Notes

If hard copies of each screen are made and put in booklet form, students have their class notes already prepared. Each lecture might contain 10-15 screens of information. If each screen is dumped to the printer, reduced, and then edited so that there are 5 or 6 screens per page for each lecture given, students will have booklets with all of the professor's presentations. Students do not need to take notes on the lecture because the hard copies of the screens are already in their possession. Students highlight only the portions of the screen dumps that they want to emphasize because most of their note taking has already been done for them. This leaves the students with more time to concentrate on the topic and learn.

Advantages for the Teacher

The teacher also benefits from using the software. The lecture has been prepared and is on the classroom disk. Preparing for a class is as simple as looking at the hard copies of the screens or running through the presentation before going to class. The software is very user friendly. Even the instructor who refuses to learn anything about computers could use the *prepared* lectures. These standardized lectures are valuable when there are many sections of the same course. Once a lecture has been written, it can be used indefinitely.

A Student Opinion Survey

Results of a student opinion survey in an Intermediate Algebra class which utilized presentation software were favorable. Students were asked to compare the class where the software was used with a comparable mathematics class where a traditional delivery was used. A class of 25 students responded as follows:

STATEMENT	YES
Did you take more notes in this class?	48%
Did your attention span increase in this class?	52%
Was this class more interesting?	76%
Did you get better grades because of the software?	34%
Did you want the software in the lab for your use?	67%

Presentation software is a viable tool for teaching mathematics. While it is not mathematical in nature like function plotters or symbolic manipulators, it does give the instructor a much-needed alternative style to teaching mathematics.

Appendix

Show Partner F/X
Computer software by Brightbill-Roberts and Company. Retails at $350. Reviewed in *PC Computing*, February, 1988: p. 77.

PC Storyboard
Computer software by IBM Corporation. No longer sold at retail but perhaps can be purchased used. Reviewed in *PC Magazine* March 10, 1987: p. 249 and December 23, 1986: p. 199.

PC Storyboard Plus
Computer software by IBM Corporation. Retails at $350. Reviewed in *PC Magazine* September 27, 1988: p. 241. PC Storyboard Plus, with a VGA graphics board and a digitized board can capture any still from a video camera.

Concorde
Computer software by Visual Communications Network. Retails at $695. Reviewed in *PC Week* October 21, 1986: p. 83.

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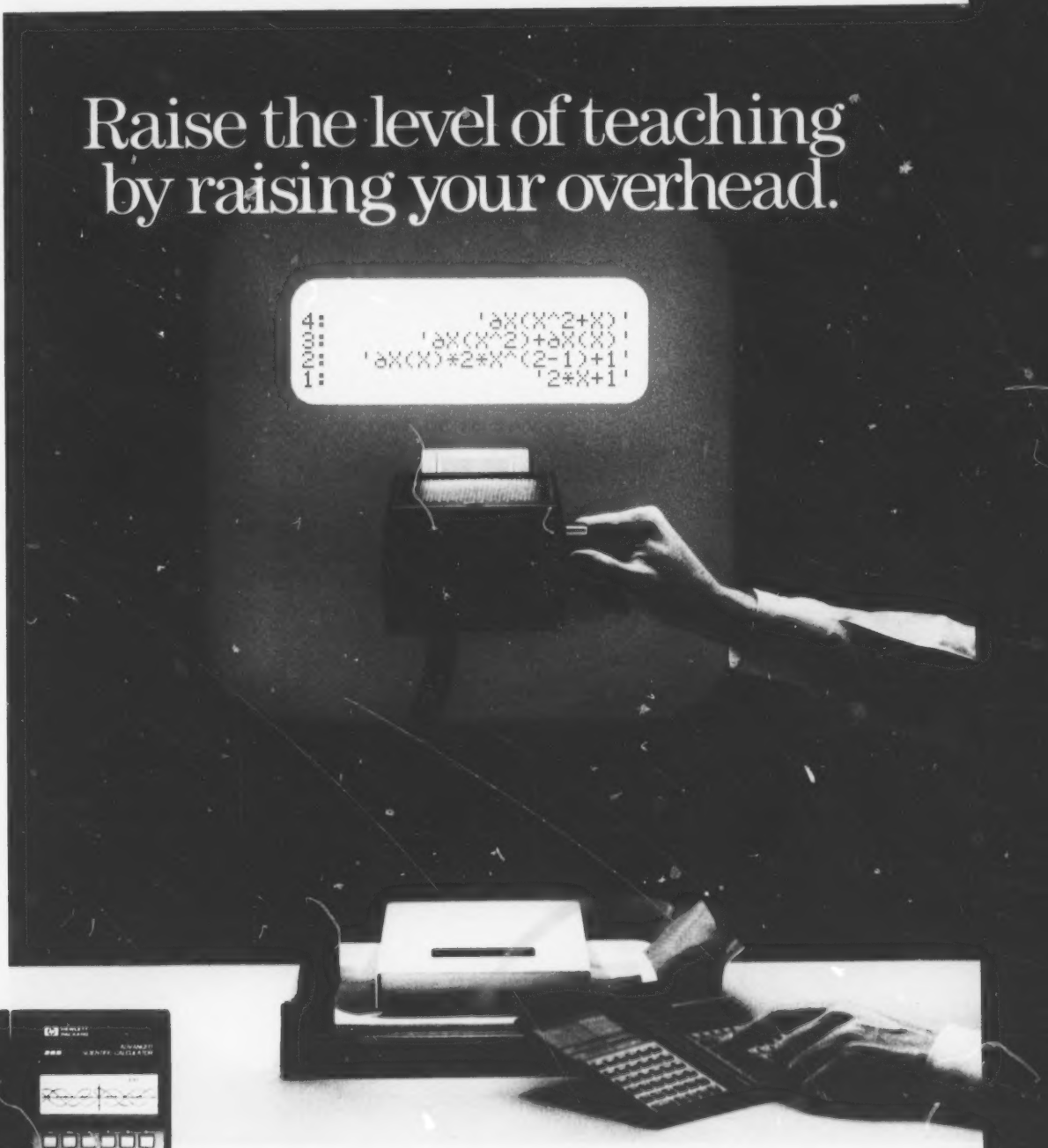

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Using Videotapes for Mathematics Instruction

Karen T. Sharp

Mott Community College, Flint, Michigan 48502

The growing number of nontraditional students who are enrolling in two-year college programs have nontraditional needs which must be met if they are to be successful. One of their most important needs is an instructional delivery system which provides for flexibility in when they study and for the length of time they study. In my opinion, videotapes or other forms of prerecorded materials can provide this flexibility. Let me explain.

The Advantages of Telecourses

Many older students have major commitments in their lives other than school. For example, many of my students have full-time jobs, families with small children, and must spend time commuting to campus. With videotaped learning, commonly called "telecourses," more flexibility is provided to these students in that lectures may be watched at home, frequently at the best possible time for the student, and the number of trips to the campus are minimized.

Videotaped lectures also provide opportunities for handicapped, under-prepared and learning disabled students to succeed more easily. For example, interpreters can be inserted into the corners of videotapes for the hearing impaired. Since students have control of the video players, entire lectures or any parts can be viewed over and over again. Similarly, students can watch portions of a lecture, then stop and take time to absorb that material before watching the remainder. On the other hand, opportunities for developmental students to progress more rapidly through mathematics courses when they need only a quick review can also be provided. While videotaped lectures are especially appropriate for what I have called the "nontraditional" student, videotapes can also be used by students enrolled in a traditional course offered on campus for supplemental learning or when they have missed lectures.

Video learning has advantages for instructors and colleges as well. Low enrollment classes can be offered every semester without danger of cancellation. Scheduling conflicts of all sorts can be resolved. Instructors can channel time, energy, and enthusiasm usually reserved for routine preparation and delivery of lectures into other endeavors, such as answering students' questions about assignments, working with students having severe difficulty with the material, curriculum development and research.

A Basic Mathematics Telecourse

My feelings regarding telelearning have been developing since I first became involved with a telecourse in BASIC computer programming on my campus three years ago. Local surveys of students in 1986 and 1987 regarding telecourses they would like to see offered showed mathematics to be their first choice. Because of my involvement with telecourses and the teaching of mathematics, I was asked by my Dean and the Director of Instructional Media Services to search for a telecourse in developmental mathematics. Since none of the commercial offerings available appeared to suit our needs, I developed a basic mathematics telecourse with the assistance of our Instructional Media Department.

I was given release time during the summer of 1988 to place the lectures for my Basic Mathematics course onto videotape and to prepare supporting materials as a prelude to offering it as a telecourse during the winter semester of 1989. This was the first telecourse produced by my college.

The mathematics telecourse is listed in the regular class schedule booklet and may be selected by any of our students. The course covers the arithmetic of whole numbers, decimals, and fractions; percentage, ratio and proportion; and a brief introduction to geometric facts, measurement systems and algebra. All telecourse students attend an on-campus orientation session at which course information is distributed along with a telecourse handbook prepared by Instructional Media Services. The information I prepared includes my name, office location, office telephone, office hours, weekly assignments which are correlated with the lesson number on the videotapes, testing information, and grading procedures. It is important to note that the first video lesson introduces the students to the textbook they will be using, its features, what is expected of them in terms of grades and

---

Karen has been a mathematics and computer science professor at Mott Community College in Flint, Michigan for seventeen years. She has been Treasurer of AMATYC for the past four years and is the 1989 recipient of the Central Region Outstanding Faculty Member Award from the Association of Community College Trustees (ACCT).

homework, and a suggested plan for success in the course.

An optional review session is held on campus prior to each examination. Tests are given in the viewing/listening center of the college library, and the students are given a choice of several days to take each examination. Night or weekend testing is available by special arrangement. Prior to taking each examination, students turn in homework.

All the videotapes for telecourses taught at Mott Community College are available for viewing at libraries throughout the county. In addition, most telecourses are broadcast over local television. The mathematics telecourse was not broadcast on television as it was thought the casual viewer would not have much interest in watching a mathematics lesson. In order to make the mathematics lectures easier for the students to view, they have the option of renting all the lecture tapes for $15.00 for the semester. If a student has a VCR at home, renting the tapes is desirable. This option has been very popular with the students, and the director of the Instructional Media Services indicates that preparing and renting the video tapes is easer and less expensive than broadcasting the videos for a telecourse. He plans to expand the rental option to other telecourses.

Telecourse Results

I supervised the teaching of Basic Mathematics to 38 students as a telecourse for the first time during the winter semester of 1989. At the same time, I was also responsible for teaching Basic Mathematics to 33 students in a traditional on-campus instructional setting. Basic Mathematics at Mott Community College is graded "S" (Satisfactory), "U" (Unsatisfactory), "W" (Withdrawal) or "I" (Incomplete). A comparison of course grades is given in Table 1.

	Telecourse	Traditional Instruction
S	21	19
U	5	6
W	3	4
I	9	4

Table 1. A comparison between course grades in Basic Mathematics using a telecourse and using traditional instruction.

The results of the comparison confirm my belief that students perform at least as well in a telecourse as they do in traditional modes of instruction. Students taking Basic Mathematics, which is developmental in nature, vary widely in mathematical abilities and study skills. Hence, granting incompletes is not uncommon in this course. A strength of the telecourse is the ease afforded students in completing course requirements after the semester is over.

Student comments regarding the Basic Mathematics telecourse have been very positive. In fact, students from other Basic Mathematics classes are checking the tapes out of the library for viewing in addition to attending their regular classes on campus.

Integrating Videotapes with Other Technology

The integration of videotapes with other types of technology such as computers and laser disk players to form an interactive learning environment is starting to occur at colleges. However, the lack of easy access to video disc technology as opposed to the extremely easy access to videotape players makes video learning feasible and more desirable at this time. Computer authoring systems can be utilized to customize instruction and activities for students. In fact, as a part of the Basic Mathematics telecourse, a fractions review and fractions examination were prepared using the computer authoring language Quest.

Summary

Others are beginning to recognize the value of videotapes for mathematics instruction. Many publishers are now producing videotapes as supplements to their textbooks. In addition, COMAP produced the mathematics video series *For All Practical Purposes* that was broadcast recently on many PBS television stations. On a more personal note, I have been given approval by my college to produce a telecourse in Beginning Algebra during the summer of 1989, with the telecourse to be offered in winter term, 1990. I am looking forward to expanding opportunities and increased student success through the use of telelearning.

Author's Note. Interested readers are invited to contact the author for more information.

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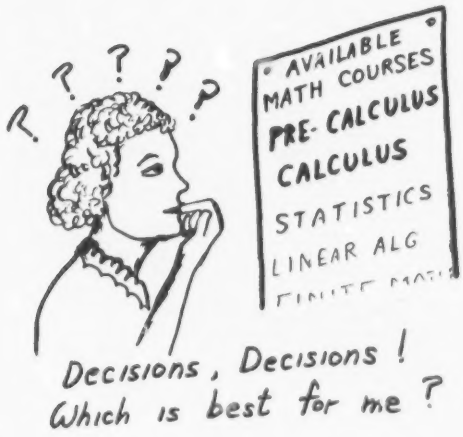
Section 5.
Placement



- What about Calculator-Based Placement Tests? – by John G. Harvey

  "In 1986 an important question was whether multiple-choice calculator-active test items could be developed, especially, for new test objectives or for higher-level knowledge. Happily, as I demonstrate later, multiple-choice calculator-active test items can be developed even when problem solving is being tested."

- It's Time to Reexamine Placement – by John H. Jenkins

  "...most of the concern [about calculus] has centered on the curriculum and the failure rate and little has been said about the initial placement of students into calculus."

  "If some way could be developed to use the first examination [in calculus] as a placement tool, then it would also measure, to a certain extent, motivation and study skills."

What About Calculator-Based Placement Tests?

John G. Harvey

University of Wisconsin, Madison, Wisconsin 53706

For more than a decade mathematicians and mathematics educators at both the school and college levels have recommended that calculators be used both during mathematics instruction and testing (College Board, 1983, 1985; Commission on Standards for School Mathematics, 1989; Conference Board of the Mathematical Sciences, 1983; Douglas, 1986; National Advisory Committee on Mathematical Education, 1975; National Council of Teachers of Mathematics (NCTM), 1980, 1986; Steen, 1987). This recommendation is summed up by the NCTM calculator position statement; that statement clearly recommends the use of calculators during testing in this concluding remark:

> The National Council of Teachers of Mathematics recommends that publishers, authors and test writers integrate the use of the calculator into their mathematics materials at all grade levels. (NCTM, 1986)

School and college mathematics teachers are using and are permitting or requiring their students to use calculators. Data gathered by the 1986 National Assessment of Educational Progress (NAEP) Mathematics Assessment (Dossey et al., 1988, p. 79) revealed that the schools attended by about 15% of third-grade students, 21% of seventh-grade students and 26% of ninth-grade students have calculators for use in mathematics classes; almost all of these students reported that their families own a calculator. In the Chicago Public Schools *all* students in the upper elementary and middle school grades are furnished either with a simple four-function calculator or a scientific calculator (Dorothy Strong, personal communication). A recent survey of college faculty (Kupin & Whittington, 1988) showed that between 62% and 97% of the respondents used calculators in college algebra, precalculus, calculus and noncalculus based statistics, calculus, discrete mathematics and linear algebra courses. However, until national tests of mathematics achievement and aptitude permit or, indeed, require students to use calculators, many teachers will continue to teach and students will continue to learn paper-and-pencil techniques that are unnecessary and inappropriate when calculators are integrated into instruction.

It is especially important that tests whose scores are used nationwide for college admission (e.g., the Scholastic Aptitude Test) or initial college mathematics placement (i.e., mathematics placement tests) be calculator-based. Otherwise, the recommended use of calculators in schools will never be fully implemented, and the growing expectation that college students know how to use calculators while solving mathematics problems will not be met. In 1986, the College Board and the Mathematical Association of America (MAA) convened a joint symposium on the use of calculators in the standardized testing of mathematics to discuss the use of calculators with standardized tests and to make recommendations that could serve as guidelines for the development of calculator-based tests (Kenelly, 1989a). It was recommended that the College Board develop calculator-based mathematics achievement tests and that the MAA develop calculator-based placement tests. Not long after the symposium the College Board's Mathematical Achievement Test Committee began development of a calculator-based version of the Mathematics Level II Achievement Test; that test is in the prototype development stage. In October 1986, the MAA's Committee on Placement Examinations (COPE), with the financial support of Texas Instruments Incorporated, began its Calculator-Based Placement Test Program (CBPTP) Project (Harvey, 1989a; Harvey & Kenelly, 1988); this project is developing six college-level placement tests and two high school prognostic (i.e., early placement) tests. As these tests are completed they will become a part of the test packet distributed to subscribers to the MAA's Placement Test Program.

Calculator-Based Placement Tests

How does a calculator-based (placement) test differ from other tests of mathematics achievement (i.e., from tests that do not require calculator use)? Some may see little difference between testing with and without calculator use. In two instances the tests used were not developed to account for calculator use though their use was permitted (Kenelly, 1989b; Leitzel & Waits, 1989).

John is Professor of Mathematics and of Curriculum and Instruction at the University of Wisconsin-Madison. He directs the MAA Calculator-Based Placement Test Program Project and the MAA Prognostic Testing Network Project. He is presently Chair of the MAA Committee on Placement Examinations and served on the AMATYC-MAA Joint Task Force on Remediation.

In both instances the test developers subsequently developed or advocated development of calculator-based tests.

In the first instance calculator use was permitted on the 1983 and 1984 College Board Advanced Placement (AP) Calculus Examination. Because of the problems generated by using calculators on calculator-independent AP calculus tests, the use of calculators on AP Calculus Examinations was abandoned (Kenelly, 1989b, p. 5). In the second instance, the Ohio Early Mathematics Placement Program (EMPT) has permitted use of calculators since 1983-84. The 1983 EMPT Program test was the same one given in 1982-83 except that one obviously calculator-dependent item was deleted from the 1982-83 test. All of the tests used were *calculator-neutral* or calculator-independent; that is, none of the test items required calculator use in order to be correctly answered.

What are the characteristics of a calculator-based test? In this paper the definitions of calculator-active test items and of calculator-based tests are:

> **A calculator-active test item** is an item that (a) contains data that can be usefully explored and manipulated using a calculator and (b) has been designed to require active calculator use.
>
> **A calculator-based mathematics test** is one that (a) tests mathematics achievement, (b) has some calculator-active test items on it and (c) has no items on it that could have been, but are not, calculator-active *except for* items that are better solved using noncalculator based techniques.

These definitions are modifications of ones I have previously given and are the ones being used by the MAA Calculator-Based Placement Test Program Project (CBPTP) test panels while developing their placement tests. Other definitions or decisions need to be made as well.

Traditional Decisions. In order to develop calculator-based placement tests many decisions must be made. These decisions include the traditional ones about the content of the test, the number of items and the format of the test items.

Only the decision about the number of items remains straightforward. This number continues to be determined by the number of objectives tested, the number of items for each objective that are needed, the amount of time required to respond to each objective and the total amount of time that can be allotted to administering the test.

The decision about the format of the calculator-active items could be a difficult one. Previously, most mathematics placement tests consisted of multiple-choice items. The use of multiple-choice items on placement tests is important for two reasons: (a) it is easier to generate parallel forms of multiple-choice items than it is for other item formats and (b) placement test scores need to be available to students and advisors almost immediately after the test is administered. At present, I know of no other test question format that can be scored as quickly, as easily or as cheaply. In 1986 an important question was whether multiple-choice calculator-active test items could be developed, especially, for new test objectives or for higher-level knowledge. Happily, as I demonstrate later, multiple-choice calculator-active test items can be developed even when problem-solving is being tested.

The remaining traditional decision about the content of the placement test could easily be confounded by whether or not only calculator-active items will be used and by whether or not students' calculator facility will be tested. These are difficult issues and were discussed by the participants in the College Board-MAA symposium; their discussion of these issues resulted in these recommendations:

1. Mathematics achievement tests should be curriculum based, and no questions should be used on them that measure only calculator skills or techniques.

2. When addressing a particular test item, an important skill is choosing when and when not to use a calculator. Consequently, not all of the questions on a calculator-based mathematics achievement test should require the use of a calculator.

These recommendations were endorsed by the CBPTP Project Policy Group and are being used while developing the MAA calculator-based placement tests.

These two symposium recommendations are certainly reasonable ones. Placement tests constructed using the first recommendation will clearly have good face validity; that is, as in the past, these new tests will only test a student's knowledge of mathematical content. Since calculator use will be required on some of the test items, the scores on calculator-based tests will also reflect a student's knowledge of calculator functionality. Thus, tests constructed using the first recommendation will probably give little evidence of why a student incorrectly responded or failed to respond to an item; that is, we will not know whether it is because he or she did not have the necessary mathematical knowledge or because he or she did not have the necessary calculator skills.

Placement tests constructed using the second recommendation will consist of items chosen from the domains of arithmetic, algebra and geometry as do present placement tests. The difference between the calculator-based tests and present ones will be that students will have to choose between using and not using their calculators. Some students, possibly the weaker ones, will not do this very well.

The implementation of both of these recommendations could be detrimental to students in some testing situations since mathematical knowledge and knowledge

of calculator functionality are both being tested and since students will have to make difficult decisions about when to use their calculators. But placement tests developed using these recommendations are advantageous since calculator-based placement tests are likely to be given only when the courses into which these tests place students will be ones where students are expected to use calculators, to have the necessary knowledge of calculator functionality and to have developed some skill in distinguishing situations where calculator use will be effective.

Calculator Decisions. Before developing a calculator-based placement test it is necessary to determine what calculators students will be permitted to use. The present spectrum of calculators ranges from simple four-function calculators to those that graph functions and have limited symbolic mathematics abilities. Presently, the least capable calculators that seem appropriate for college-level placement testing are the scientific, nonprogrammable ones; the calculators less capable than these do not have the needed functionality. In addition, scientific, nonprogrammable calculators are readily available and inexpensive, and many students already have one.

The emergence of graphics and symbolic mathematics calculators complicates the development of calculator-based placement tests since their added capabilities can make some test items calculator-dependent or could give undue advantages to students having them. Since most students will be permitted to use their own calculators during placement testing, the capabilities of their calculators must be considered when developing calculator-based placement tests. An item may become calculator-dependent or its level may drop when a more capable calculator is used (Harvey, 1989b). However, these calculators may not be an important factor yet in placement testing since few entering college students have them. More important, as the number of students having these calculators increases, it will probably be necessary to specify that the least capable calculator that can be used during placement testing is one of the more capable machines.

Calculator-Active Items

Persons developing calculator-active items need to be very knowledgeable about the functionality of the calculator that students will use while responding to that item. They must lay aside the paper-and-pencil techniques they have learned for solving the problem posed in the item stem and think like a student who has learned calculator-based techniques for solving the problem. One way for the mathematics teachers to do this is to actively integrate calculator use into their teaching; the integration of even scientific, nonprogrammable calculators into instruction may cause changes in pedagogy (Harvey, in press) and should certainly better prepare the teacher to develop calculator-active test items and students to take tests successfully that contain calculator-active items. It is clear to me that if calculator-based tests are going to test students' knowledge of mathematics accurately, then students must have a good knowledge of calculator functionality.

Using the perspective of a calculator-using student, an item writer must also determine that the proposed item tests the mathematical objective(s) intended for it; the members of the CBPTP Program test panels have discovered that sometimes the objectives tested by items will change when calculator-based techniques instead of paper-and-pencil techniques are used. If a multiple-choice item is being developed, then the mathematical errors that students make should still determine what the incorrect choices are; the incorrect choices listed should reflect the mathematical errors made by students when using calculators since sometimes the incorrect answer derived using paper-and-pencil techniques and calculator-based techniques will differ.

A calculator-active item requires active calculator usage to respond to it. Thus, one approach to developing a calculator-active item would be simply to change some or all of the common fractions in an existing item into decimal numbers. This is a strategy that can be applied in some instances though both of the criteria for a calculator-active item may not be completely satisfied. In Figure 1 is an example of an item from the MAA Calculator-Based Arithmetic and Skills (CB-A-S) Test that meets one but not both of the criteria for a calculator-active item even though all of choices are decimals and the number in the item stem is a common fraction.

The decimal representation of 5/40 is

(A) 0.08 (B) 0.125 (C) 1.25 (D) 5.40 (E) 8

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Figure 1. Sample calculator-active item requiring fraction to decimal conversion.

For the calculator-using student this is clearly an item that can be actively explored with a calculator. Students responding to this item do have the option of dividing the denominator into the numerator of the fraction, and so the second criterion for a calculator-active item is not met. The item in Figure 2 from the CB-A-S Test meets both criteria. It tests a student's knowledge of exponentiation and order of operations.

Mathematics faculty, graduate students and possibly, calculus students could respond to this item by using the Binomial Theorem to expand the expression and then computing the value of each addend. However, students taking the CB-A-S Test are ones who will be placed into a remedial arithmetic course or a basic algebra course and do not know the Binomial Theorem. Therefore, for

The best approximation of $(1+1/6)^4$ correct to 4 decimal places is

(A) 1.0008 (B) 1.1667 (C) 1.8526 (D) 2.1614 (E) 4.6667

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Figure 2. Placement test item satisfying both criteria for a calculator-active item.

these students the item requires calculator use. Each of this item's incorrect responses is based upon mathematical errors that students might make.

The first two examples of calculator-active items are from arithmetic. Figure 3 shows an item that tests a student's ability to solve an equation. It is anticipated that this item will be on the MAA Calculator-Based Algebra Test.

If $(x + 8)^3 = 14$, then x is best approximated by

(A) -7.93 (B) -5.59 (C) 1.82 (D) 6 (E) 7.93

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Figure 3. Calculator-active equation solving item.

The MAA Calculus-Readiness (CR) Test is used to determine whether students should or should not be enrolled in a first-semester calculus course. Thus, it assesses higher-level knowledge. Two examples of calculator-active items on the MAA Calculator-Based Calculus Readiness (CB-CR) Test are given in Figures 4 and 5.

The number approached by the sequence of numbers $(3/2)^4$, $(4/3)^6$, $(5/4)^8$, ... , $((n+1)/n)^{2n}$, ... is best approximated by

(A) 1 (B) 2.178 (C) 6.192 (D) 7.389 (E) No finite number

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Figure 4. CB-CR example item.

MAA Calculator-Based Placement Tests

The MAA Placement Test (PT) Program was started in 1977 to provide valid, reliable mathematics placement tests at a reasonable cost to postsecondary institutions. At present about 400 two- and four-year colleges and universities subscribe to the PT Program; approximately one-third of these subscribers are two-year colleges andtechnical institutes. The PT Program test packet consists of six placement tests; the present tests range from a test of arithmetic and basic skills to a test of calculus readiness. There are three algebra tests: basic algebra, algebra and advanced algebra. In addition, there is a

The sum of the areas of the rectangles shaded in the figure below is best approximated by

(A) 0.131 (B) 0.194 (C) 0.538 (D) 1.311 (E) 1.944

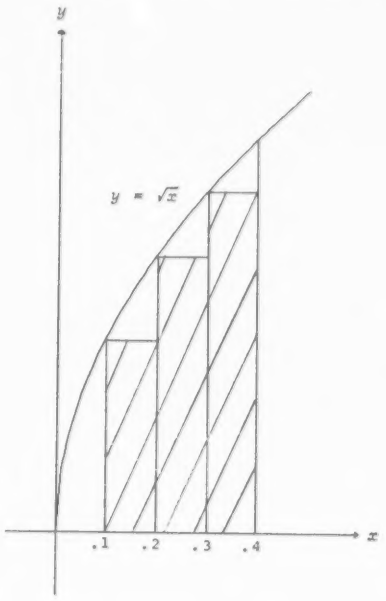

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Figure 5. Area approximation calculator-active item.

trigonometry test that includes items testing knowledge of elementary functions.

When the MAA CBPTP Program completes its tasks, there will be six MAA calculator-based placement tests; these tests will test the same range of content as do the existing MAA tests. A scientific, nonprogrammable calculator will be required to take any one of the calculator-based test. On each test at least 25% of the items will be calculator-active ones. Two of these new tests, the Calculator-Based Arithmetic and Skills Test and the Calculator-Based Calculus Readiness Test, will be included in the PT Program test packet beginning in mid-1989. The Calculator-Based Basic Algebra and Calculator-Based Algebra tests will be in the test packet beginning in mid-1990. The remaining two tests – of advanced algebra and trigonometry – will be introduced in 1991. In addition, it is planned that an MAA Note will completely describe the development and characteristics of each of the MAA calculator-based tests.

Conclusion

The use of calculator-based placement tests should help to achieve the goal of effectively integrating calculators into mathematics instruction and learning.

Presently, the placement tests being developed are ones that require students to use scientific calculators. To anticipate and to assist in integrating graphics and symbolic mathematics calculators into the college mathematics curriculum, we will need to develop additional placement tests. As a result, we will need to continue to study ways to generate valid calculator-active mathematics items.

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It's Time to Reexamine Placement

John H. Jenkins

Embry-Riddle Aeronautical University, Prescott, AZ 86301

During the last three years there have been two national conferences as well as numerous presentations at mathematics meetings on the calculus "crisis." However, most of the concern has centered on the curriculum and the failure rate, and little has been said about initial placement of students into calculus. It is time to carefully examine the placement program.

The purpose of this article is to describe the new placement program used by the Prescott Campus of Embry-Riddle Aeronautical University. It is an attempt to measure more than just prerequisite mathematics skills as criteria for placement into Calculus I.

Background

Embry-Riddle Aeronautical University is the world's largest and oldest totally aviation-oriented, accredited university. A private, independent, co-educational institution founded in 1926, Embry-Riddle has an enrollment of about 13,000 full-time and part-time students. ERAU offers academic degrees through the Master's level from its disciplines of engineering, avionics, aviation maintenance, flight, and business administration. ERAU has residence campuses in Daytona Beach, Florida and Prescott, Arizona and over 80 off campus centers at military bases worldwide. The Prescott campus was established in 1978 and today has an enrollment of over 1600 students. This article will focus on that campus.

In 1987 there were 121 entering students who were candidates for Calculus I because they were majoring in engineering or computer sciences. Of these, only 22% passed Calculus I; the rest were either placed by our placement procedure into College Algebra, or failed or withdrew from Calculus I.

In 1988, 45% of the 138 entering calculus candidates passed Calculus I. This major change in just one year is attributed to a placement program that is based on performance, rather than on a single multiple choice test that is administered before school begins.

John has been a member of the mathematics department at ERAU for the past 11 years. He received his BA in mathematics from the University of Tennessee at Chattanooga in 1971 and his MAT in mathematics from the University of Florida in 1974.

The incoming freshman classes in 1987 and 1988 were almost identical as Table 1 indicates.

TABLE 1
Incoming Class

	Average Score		HS GPA	High School Rank	
	SAT-M	ACT-M	≥3.00	Top 20%	Below 50%
1987	570	25	70%	47%	11%
1988	580	27	76%	60%	4%

In 1987, the only criteria for placement into Calculus I was a score of 12 or better on the Calculus Readiness test from the MAA Placement Testing Program. Of the 121 students who took this test, only 49 scored twelve or above and were placed into Calculus I. Of these 49, only 27 achieved a grade of C or better in the course. However, 44 had completed three or more years of high school mathematics and 13 of the 49 had some exposure to calculus in high school. These students had good SAT/ACT scores and were close to the top of their high school classes. Background was not the problem.

Was there some way to tell how well a student was going to perform based on some data available at the beginning of the course? At the end of the fall semester of 1987, a study was undertaken to see if a better pass rate could be achieved if a different placement procedure was followed. Several correlation tests between various sets of data were calculated to determine if there was any pattern. (See Table 2). The highest correlation was 0.47. This was between the HS GPA and the final grade in the course. The same correlation tests were run using the results from the first examination. The highest correlation, 0.52, was between the ACT M and the first examination.

The MAA Calculus Readiness Test was a poor predictor of success in Calculus I. Correlation between the CR Test and the first examination was -0.05 and was only a 0.20 with the final grade.

The final correlation test performed was between the first examination and the final course grade. The correlation here was 0.69. (The correlation between the first examination and the final percentage instead of the final

grade was 0.85.) With this result, we felt that the first examination was a better predictor of success in the course than the Calculus Readiness Test.

TABLE 2
Correlation for Students in Calculus I

	HS MTH GPA	HS GPA	HS RANK	SAT-M	ACT-M	MAA/CR
& First Exam						
1987	0.24	0.37	0.04	0.36	0.52	−0.05
1988	0.20	0.11	0.35	0.18	0.27	NA
& Final Grade						
1987	0.31	0.47	0.30	0.56	0.31	0.20
1988	0.18	0.20	0.28	0.21	0.21	NA
First Exam & Final Grade		1987 = 0.69		1988 = 0.45		

No placement test given before classes start can measure motivation. If some way could be developed to use the first examination as a placement tool, then it would also measure, to a certain extent, motivation and study skills.

The New Placement Procedure

About 200 students were expected in College Algebra and Calculus I for the fall semester 1988. This required six sections of freshman mathematics. The schedule was set up so there were two time blocks for freshman mathematics. Three sections of Calculus I were put into each time block. Three faculty members were assigned to teach these six sections.

An identical curriculum composed of precalculus review topics commonly found in calculus textbooks was taught to each section for the first nine sessions. Each of the three faculty members taught three lectures to each group. At the end of the nine lectures a common examination was given to all of the students. Each faculty member made up one-third of this examination and graded his own portion.

A two week algebra review had always been a part of the Calculus I curriculum; therefore, to implement this program, material was not being deleted from the course.

Based on the results of this common exam, the students were placed into either Calculus I or College Algebra. The students' schedules were rearranged (drop/add) at this point. A specific teacher was assigned to remain with each class. The team-teaching aspect of the proposal ceased after the first examination. Sections of Calculus I ran normally for the remainder of the semester. Sections of College Algebra that were started at this point met four days a week for the remaining eleven weeks. This gave the students the proper number of class hours for a three-credit course.

Benefits

If some students do poorly in Calculus I because they are weak in algebra, then they will do poorly on the first test and be given a second chance to succeed by being placed properly into College Algebra. These students will have experienced the pace and expectations associated with Calculus I and will hopefully work harder in College Algebra to prepare for calculus. On the other hand students who do poorly because they lack motivation will hopefully study from the beginning of the course because they know they have a placement test looming three weeks into the semester.

Currently, those students who receive a D, F, or W in Calculus I retake Calculus I in their second semester without curing their problem of weak algebra. When students are counseled to take College Algebra after doing poorly in Calculus I, the typical response is, "I can't do that. I'll get too far behind in my program." This program offers students the opportunity to cure their algebra problems during the first semester without receiving poor grades on their transcripts.

Outcome

In the fall of 1988, 138 new students entered in engineering and computer science (54 returning students had already preregistered). On the day before registration, the placement system was explained to the new students. During the first nine class sessions we continued to emphasize the importance of the placement test. During this time, 20 student elected to place themselves into College Algebra. Thirty-three of the students had a "lock" on a seat in Calculus I because they were either repeating it or had passed College Algebra. Of the remaining 139 students, 90 scored 60% or better on the placement test and were placed into Calculus I. Of the 49 who were placed into College Algebra only four challenged this placement. They were allowed to register for Calculus I. (None of these four passed Calculus I.) We sent letters to all of the students who had received a score in the 60's, suggesting that they consider taking College Algebra instead. None of these students elected to take College Algebra.

The pass rate for those students placed into Calculus I went from 55% in 1987 to 69% in 1988, but the success rate of the incoming students went from 22% to 45%. (See Table 3.) The professors who taught College Algebra in 1987 felt that there were a number of students in their classes who should have been in Calculus I. The professor who taught College Algebra in 1988 did not have the same feeling.

If we repeat this program, we would start one section of College Algebra at the beginning of the semester. There were enough students who knew they were going to take College Algebra regardless of placement. These students loafed the first two weeks and were able to

TABLE 3
Pass Rate and Attrition

	# Incoming Students	Algebra (pass %)	Calculus (pass %)	% Total pass Calc.	Attrition after 1 sem.
1987	121	72 (83%)	49 (55%)	22%	15%
1988	138	48 (67%)	90 (69%)	45%	15%

develop some bad study habits. A four day a week program is too intense for some of these students.

Conclusion

Although this system worked better than a placement test, we are still not satisfied with the results. Success in Calculus I depends on two factors: preparation and motivation. The student is the only one who can determine motivation. Preparation is better measured by someone who has had more contact with the student.

In May we will send our course descriptions to all of our applicants and ask them to confer with their high school mathematics teachers to determine the course for which they should register. Placement in Calculus I will be checked by the student's score on the first test. If a student does poorly on the first test, he/she will be given the option of moving to a section of College Algebra. If it shows that students are accurately placing themselves, then ERAU, Prescott, will get out of the placement business.

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